A SCHOOL GEOMETRY

PARTS III. IV. AND V.

(Containing the substance of Euclid Books II., III., IV. and VI., with Additional Theorems and Examples)

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PREFACE.

The present work provides a course of Elementary Geometry based on the recommendations of the Mathematical Association and on the schedule recently proposed and adopted at Cambridge.

The principles which governed these proposals have been confirmed by the issue of revised schedules for all the more important Examinations, and they are now so generally accepted by teachers that they need no discussion here. It is enough to note the following points :

(i) We agree that a pupil should gain his first geometrical ideas from a short preliminary course of a practical and experimental character. A suitable introduction to the present book would consist of Easy Exercises in Drawing to illustrate the subject matter of the Definitions; Measurements of Lines and Angles; Use of Compasses and Protractor; Problems on Bisection, Perpendiculars, and Parallels; Use of Set Squares; The Construction of Triangles and Quadrilaterals. These problems should be accompanied by informal explanation, and the results verified by measurement. Concurrently, there should be a series of exercises in Drawing and Measurement designed to lead inductively to the more important Theorems of Part I. [Euc. I. 1-34].* While strongly advocating some such introductory lessons, we may point out that our book, as far as it goes, is complete in itself, and from the first is illustrated by numerical and graphical examples of the easiest types. Thus, throughout the whole work, a graphical and experimental course is provided side by side with the usual deductive exercises.

(ii) Theorems and Problems are arranged in separate but parallel courses, intended to be studied *pari passu*. This arrangement is made possible by the use, now generally sanctioned, of *Hypothetical Constructions*. These, before being employed in the text, are carefully specified, and referred to the Axioms on which they depend.

* Such an introductory course is now furnished by our *Lessons in Experimental and Practical Geometry*.

(iii) The subject is placed on the basis of *Commensurable Magnitudes.* By this means, certain difficulties which are wholly beyond the grasp of a young learner are postponed, and a wide field of graphical and numerical illustration is opened. Moreover the fundamental Theorems on Areas (hardly less than those on Proportion) may thus be reduced in number, greatly simplified, and brought into line with practical applications.

(iv) An attempt has been made to curtail the excessive body of text which the demands of Examinations have hitherto forced as "bookwork" on a beginner's memory. Even of the Theorems here given a certain number (which we have distinguished with an asterisk) might be omitted or postponed at the discretion of the teacher. And the formal propositions for which—as such—teacher and pupil are held responsible, might perhaps be still further limited to those which make the landmarks of Elementary Geometry. Time so gained should be used in getting the pupil to *apply* his knowledge; and the working of examples should be made as important a part of a lesson in Geometry as it is so considered in Arithmetic and Algebra.

Though we have not always followed Euclid's order of Propositions, we think it desirable for the present, in regard to the subject-matter of Euclid Book I. to preserve the essentials of his logical sequence. Our departure from Euclid's treatment of Areas has already been mentioned; the only other important divergence in this section of the work is the position of I. 26 (Theorem 17), which we place after I. 32 (Theorem 16), thus getting rid of the tedious and uninstructive *Second Case.* In subsequent Parts a freer treatment in respect of logical order has been followed.

As regards the presentment of the propositions, we have constantly kept in mind the needs of that large class of students, who, without special aptitude for mathematical study, and under no necessity for acquiring technical knowledge, may and do derive real intellectual advantage from lessons in pure deductive reasoning. Nothing has as yet been devised as effective for this purpose as the Euclidean form of proof; and in our opinion no excuse is needed for treating the earlier proofs with that fulness which we have always found necessary in our experience as teachers.

The examples are numerous and for the most part easy. They have been very carefully arranged, and are distributed throughout the text in immediate connection with the propositions on which they depend. A special feature is the large number of examples involving graphical or numerical work. The answers to these have been printed on perforated pages, so that they may easily be removed if it is found that access to numerical results is a source of temptation in examples involving measurement.

We are indebted to several friends for advice and suggestions. In particular we wish to express our thanks to Mr. H. C. Playne and Mr. H. C. Beaven of Clifton College for the valuable assistance they have rendered in reading the proof sheets and checking the answers to some of the numerical exercises.

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November, 1903.

PREFATORY NOTE TO THE THIRD EDITION.

In the present edition some further steps have been taken towards the curtailment of bookwork by reducing certain less important propositions (e.g. Euclid I. 22, 43, 44) to the rank of exercises. Room has thus been found for more numerical and graphical exercises, and experimental work such as that leading to the Theorem of Pythagoras.

Theorem 22 (page 62), in the shape recommended in the Cambridge Schedule, replaces the equivalent proposition given as *Additional Theorem A* (page 60) in previous editions.

In the case of a few problems (e.g. Problems 23, 28, 29) it has been thought more instructive to justify the construction by a preliminary *analysis* than by the usual formal proof.

H. S. HALL.
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March, 1904.

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THE CIRCLE.

Definitions and First Principles.

1. A circle is a plane figure contained by a line traced out by a point which moves so that its distance from a certain fixed point is always the same.

The fixed point is called the centre, and the bounding line is called the circumference.

Note. According to this definition the term circle strictly applies to the *figure* contained by the circumference; it is often used however for the circumference itself when no confusion is likely to arise.

2. A radius of a circle is a straight line drawn from the centre to the circumference. It follows that all radii of a circle are equal.

3. A diameter of a circle is a straight line drawn through the centre and terminated both ways by the circumference.

4. A semi-circle is the figure bounded by a diameter of a circle and the part of the circumference cut off by the diameter.

It will be proved on page 142 that a diameter divides a circle into two identically equal parts.

5. Circles that have the same centre are said to be concentric.

From these definitions we draw the following inferences:

(i) A circle is a *closed* curve; so that if the circumference is crossed by a straight line, this line if produced will cross the circumference at a second point.

(ii) The distance of a point from the centre of a circle is greater or less than the radius according as the point is without or within the circumference.

(iii) A point is outside or inside a circle according as its distance from the centre is greater or less than the radius.

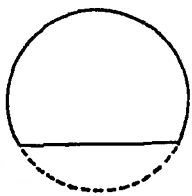
(iv) Circles of equal radii are identically equal. For by superposition of one centre on the other the circumferences must coincide at every point.

(v) Concentric circles of unequal radii cannot intersect, for the distance from the centre of every point on the smaller circle is less than the radius of the larger.

(vi) If the circumferences of two circles have a common point they cannot have the same centre, unless they coincide altogether.

6. An arc of a circle is any part of the circumference.

7. A chord of a circle is a straight line joining any two points on the circumference.



NOTE. From these definitions it may be seen that a chord of a circle, which does not pass through the centre, divides the circumference into two unequal arcs; of these, the greater is called the **major arc**, and the less the **minor arc**. Thus the major arc is *greater*, and the minor arc *less* than the semi-circumference.

The major and minor arcs, into which a circumference is divided by a chord, are said to be **conjugate** to one another.

SYMMETRY.

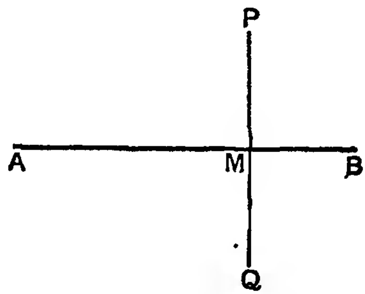
Some elementary properties of circles are easily proved by considerations of symmetry. For convenience the definition given on page 21 is here repeated.

DEFINITION 1. A figure is said to be **symmetrical about a line** when, on being folded about that line, the parts of the figure on each side of it can be brought into coincidence.

The straight line is called an **axis of symmetry**.

That this may be possible, it is clear that the two parts of the figure must have the same size and shape, and must be similarly placed with regard to the axis.

DEFINITION 2. Let AB be a straight line and P a point outside it.



From P draw PM perp. to AB, and produce it to Q, making MQ equal to PM.

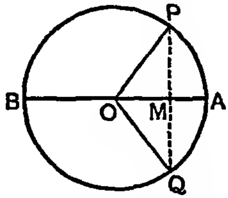
Then if the figure is folded about AB, the point P may be made to coincide with Q, for the $\angle AMP =$ the $\angle AMQ$, and $MP = MQ$.

The points P and Q are said to be **symmetrically opposite** with regard to the axis AB, and each point is said to be the image of the other in the axis.

NOTE. A point and its image are equidistant from every point on the axis. See Prob. 14, page 91.

SOME SYMMETRICAL PROPERTIES OF CIRCLES.

I. *A circle is symmetrical about any diameter.*



Let APBQ be a circle of which O is the centre, and AB any diameter.

It is required to prove that the circle is symmetrical about AB.

Proof. Let OP and OQ be two radii making any equal $\angle$s AOP, AOQ on opposite sides of OA.

Then if the figure is folded about AB, OP may be made to fall along OQ, since the $\angle$AOP = the $\angle$AOQ.

And thus P will coincide with Q, since OP = OQ.

Thus every point in the arc APB must coincide with some point in the arc AQB; that is, the two parts of the circumference on each side of AB can be made to coincide.

$\therefore$ the circle is symmetrical about the diameter AB.

COROLLARY. If PQ is drawn cutting AB at M, then on folding the figure about AB, since P falls on Q, MP will coincide with MQ,

$\therefore$ MP = MQ;

and the $\angle$OMP will coincide with the $\angle$OMQ,

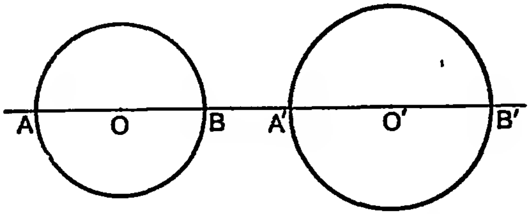
$\therefore$ these angles, being adjacent, are rt. $\angle$s;

$\therefore$ the points P and Q are symmetrically opposite with regard to AB.

Hence, conversely, *if a circle passes through a given point* P, *it also passes through the symmetrically opposite point with regard to any diameter.*

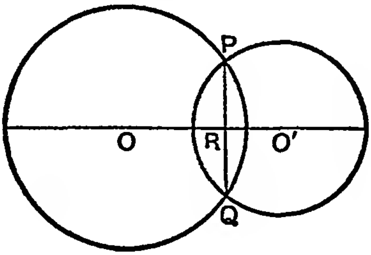
DEFINITION. The straight line passing through the centres of two circles is called the line of centres.

II. *Two circles are divided symmetrically by their line of centres.*



Let O, O′ be the centres of two circles, and let the st. line through O, O′ cut the O^{ces} at A, B and A′, B′. Then AB and A′B′ are diameters and therefore axes of symmetry of their respective circles. That is, the line of centres divides each circle symmetrically.

III. *If two circles cut at one point, they must also cut at a second point; and the common chord is bisected at right angles by the line of centres.*



Let the circles whose centres are O, O′ cut at the point P.

Draw PR perp. to OO′, and produce it to Q, so that RQ = RP.

Then P and Q are symmetrically opposite points with regard to the line of centres OO′;

∴ since P is on the O^{ce} of both circles, it follows that Q is also on the O^{ce} of both. [I. Cor.]

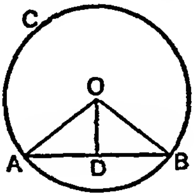
And, by construction, the common chord PQ is bisected at right angles by OO′.

ON CHORDS.

THEOREM 31. [Euclid III. 3.]

If a straight line drawn from the centre of a circle bisects a chord which does not pass through the centre, it cuts the chord at right angles.

Conversely, if it cuts the chord at right angles, it bisects it.



Let ABC be a circle whose centre is O; and let OD bisect a chord AB which does not pass through the centre.

It is required to prove that OD *is perp. to* AB.

Join OA, OB.

Proof. Then in the $\triangle^s$ ADO, BDO,

because $\begin{cases} \text{AD} = \text{BD, by hypothesis,} \\ \text{OD is common,} \\ \text{and OA} = \text{OB, being radii of the circle;} \end{cases}$

$\therefore$ the $\angle$ ADO = the $\angle$ BDO; *Theor.* 7.

and these are adjacent angles,

$\therefore$ OD is perp. to AB. Q.E.D.

Conversely. Let OD be perp. to the chord AB.

It is required to prove that OD *bisects* AB.

Proof. In the $\triangle^s$ ODA, ODB,

because $\begin{cases} \text{the } \angle^s \text{ ODA, ODB are } \textit{right angles,} \\ \text{the hypotenuse OA} = \text{the hypotenuse OB,} \\ \text{and OD is common;} \end{cases}$

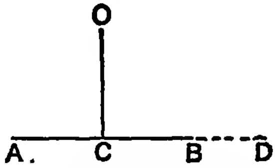
$\therefore$ DA = DB; *Theor.* 18.

that is, OD bisects AB at D. Q.E.D.

COROLLARY 1. *The straight line which bisects a chord at right angles passes through the centre.*

COROLLARY 2. *A straight line cannot meet a circle at more than two points.*

For suppose a st. line meets a circle whose centre is O at the points A and B.



Draw OC perp. to AB.

Then AC = CB.

Now if the circle were to cut AB in a third point D, AC would also be equal to CD, which is impossible.

COROLLARY 3. *A chord of a circle lies wholly within it.*

EXERCISES.

(Numerical and Graphical.)

1. In the figure of Theorem 31, if AB = 8 cm., and OD = 3 cm., find OB. Draw the figure, and verify your result by measurement.

2. Calculate the length of a chord which stands at a distance 5″ from the centre of a circle whose radius is 13″.

3. In a circle of 1″ radius draw two chords 1·6″ and 1·2″ in length. Calculate and measure the distance of each from the centre.

4. Draw a circle whose diameter is 8·0 cm. and place in it a chord 6·0 cm. in length. Calculate to the nearest millimetre the distance of the chord from the centre; and verify your result by measurement.

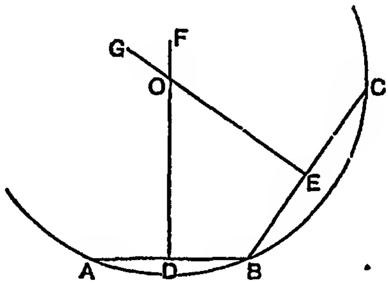
5. Find the distance from the centre of a chord 5 ft. 10 in. in length in a circle whose diameter is 2 yds. 2 in. Verify the result graphically by drawing a figure in which 1 cm. represents 10″.

6. AB is a chord 2·4″ long in a circle whose centre is O and whose radius is 1·3″; find the area of the triangle OAB in square inches.

7. Two points P and Q are 3″ apart. Draw a circle with radius 1·7″ to pass through P and Q. Calculate the distance of its centre from the chord PQ, and verify by measurement.

THEOREM 32.

One circle, and only one, can pass through any three points not in the same straight line.



Let A, B, C be three points not in the same straight line.

It is required to prove that one circle, and only one, can pass through A, B, *and* C.

Join AB, BC.

Let AB and BC be bisected at right angles by the lines DF, EG.

Then since AB and BC are not in the same st. line, DF and EG are not par[l].

Let DF and EG meet in O.

Proof. Because DF bisects AB at right angles,

∴ every point on DF is equidistant from A and B.

Prob. 14.

Similarly every point on EG is equidistant from B and C.

∴ O, the only point common to DF and EG, is equidistant from A, B, and C;

and there is no other point equidistant from A, B, and C.

∴ a circle having its centre at O and radius OA will pass through B and C; and this is the only circle which will pass through the three given points. Q.E.D.

COROLLARY 1. *The size and position of a circle are fully determined if it is known to pass through three given points;* for then the position of the centre and length of the radius can be found.

COROLLARY 2. *Two circles cannot cut one another in more than two points without coinciding entirely;* for if they cut at three points they would have the same centre and radius.

HYPOTHETICAL CONSTRUCTION. *From Theorem 32 it appears that we may suppose a circle to be drawn through any three points not in the same straight line.*

For example, a circle can be assumed to pass through the vertices of any triangle.

DEFINITION. The circle which passes through the vertices of a triangle is called its circum-circle, and is said to be circumscribed about the triangle. The centre of the circle is called the circum-centre of the triangle, and the radius is called the circum-radius.

EXERCISES ON THEOREMS 31 AND 32.

(*Theoretical.*)

1. The parts of a straight line intercepted between the circumferences of two concentric circles are equal.

2. Two circles, whose centres are at A and B, intersect at C, D; and M is the middle point of the common chord. Shew that AM and BM are in the same straight line. (i)

Hence prove that the line of centres bisects the common chord at right angles.

3. AB, AC are two equal chords of a circle; shew that the straight line which bisects the angle BAC passes through the centre.

4. *Find the locus of the centres of all circles which pass through two given points.*

5. *Describe a circle that shall pass through two given points and have its centre in a given straight line.*

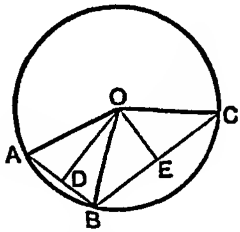
When is this impossible?

6. *Describe a circle of given radius to pass through two given points.*

When is this impossible?

*Theorem 33. [Euclid III. 9.]

If from a point within a circle more than two equal straight lines can be drawn to the circumference, that point is the centre of the circle.



Let ABC be a circle, and O a point within it from which more than two equal st. lines are drawn to the O^ce, namely OA, OB, OC.

It is required to prove that O *is the centre of the circle* ABC.

Join AB, BC.

Let D and E be the middle points of AB and BC respectively.

Join OD, OE.

Proof. In the $\triangle^s$ ODA, ODB,

because $\begin{cases} DA = DB, \\ DO \text{ is common,} \\ \text{and } OA = OB, \text{ by hypothesis;} \end{cases}$

$\therefore$ the $\angle$ ODA = the $\angle$ ODB; *Theor.* 7.

$\therefore$ these angles, being adjacent, are rt. $\angle^s$.

Hence DO bisects the chord AB at right angles, and therefore passes through the centre. *Theor.* 31, *Cor.* 1.

Similarly it may be shewn that EO passes through the centre.

$\therefore$ O, which is the only point common to DO and EO, must be the centre. Q.E.D.

EXERCISES ON CHORDS.

(*Numerical and Graphical.*)

1. AB and BC are lines at right angles, and their lengths are 1·6″ and 3·0″ respectively. Draw the circle through the points A, B, and C; find the length of its radius, and verify your result by measurement.

2. Draw a circle in which a chord 6 cm. in length stands at a distance of 3 cm. from the centre.

Calculate (to the nearest millimetre) the length of the radius, and verify your result by measurement.

3. Draw a circle on a diameter of 8 cm., and place in it a chord equal to the radius.

Calculate (to the nearest millimetre) the distance of the chord from the centre, and verify by measurement.

4. Two circles, whose radii are respectively 26 inches and 25 inches, intersect at two points which are 4 feet apart. Find the distance between their centres.

Draw the figure (scale 1 cm. to 10″), and verify your result by measurement.

5. Two parallel chords of a circle whose diameter is 13″ are respectively 5″ and 12″ in length: shew that the distance between them is either 8·5″ or 3·5″.

6. Two parallel chords of a circle on the same side of the centre are 6 cm. and 8 cm. in length respectively, and the perpendicular distance between them is 1 cm. Calculate and measure the radius.

7. Shew on squared paper that if a circle has its centre at *any* point on the x-axis and passes through the point (6, 5), it also passes through the point (6, −5). [See page 132.]

(*Theoretical.*)

8. The line joining the middle points of two parallel chords of a circle passes through the centre.

9. *Find the locus of the middle points of parallel chords in a circle.*

10. Two intersecting chords of a circle cannot bisect each other unless each is a diameter.

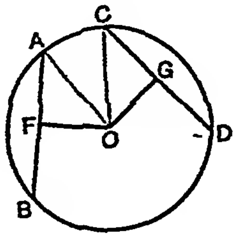
11. If a parallelogram can be inscribed in a circle, the point of intersection of its diagonals must be at the centre of the circle.

12. Shew that rectangles are the only parallelograms that can be inscribed in a circle.

THEOREM 34. [Euclid III. 14.]

Equal chords of a circle are equidistant from the centre.

Conversely, chords which are equidistant from the centre are equal.



Let AB, CD be chords of a circle whose centre is O, and let OF, OG be perpendiculars on them from O.

First. Let AB = CD.

It is required to prove that AB *and* CD *are equidistant from* O.

Join OA, OC.

Proof. Because OF is perp. to the chord AB,

∴ OF bisects AB; *Theor.* 31.

∴ AF is half of AB.

Similarly CG is half of CD.

But, by hypothesis, AB = CD,

∴ AF = CG.

Now in the Δ^s OFA, OGC,

because $\begin{cases}\text{the } \angle^s \text{ OFA, OGC are } \textit{right angles}, \\ \text{the hypotenuse OA} = \text{the hypotenuse OC}, \\ \text{and AF} = \text{CG};\end{cases}$

∴ the triangles are equal in all respects; *Theor.* 18.

so that OF = OG;

that is, AB and CD are equidistant from O.

Q.E.D.

Conversely. Let $OF = OG$.

It is required to prove that $AB = CD$.

Proof. As before it may be shewn that AF is half of AB, and CG half of CD.

Then in the $\triangle^s$ OFA, OGC,

because $\begin{cases} \text{the } \angle^s \text{ OFA, OGC are } \textit{right angles}, \\ \text{the hypotenuse OA} = \text{the hypotenuse OC}, \\ \text{and OF} = \text{OG}; \end{cases}$

$\therefore$ AF = CG; *Theor.* 18.

$\therefore$ the doubles of these are equal;

that is, AB = CD.

Q.E.D.

EXERCISES.

(Theoretical.)

1. *Find the locus of the middle points of equal chords of a circle.*

2. If two chords of a circle cut one another, and make equal angles with the straight line which joins their point of intersection to the centre, they are equal.

3. If two equal chords of a circle intersect, shew that the segments of the one are equal respectively to the segments of the other.

4. In a given circle draw a chord which shall be equal to one given straight line (not greater than the diameter) and parallel to another.

5. PQ is a fixed chord in a circle, and AB is any diameter: shew that the sum or difference of the perpendiculars let fall from A and B on PQ is constant, that is, the same for all positions of AB.

[See Ex. 9, p. 65.]

(Graphical.)

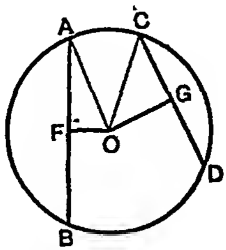
6. In a circle of radius 4·1 cm. any number of chords are drawn each 1·8 cm. in length. Shew that the middle points of these chords all lie on a circle. Calculate and measure the length of its radius, and draw the circle.

7. The centres of two circles are 4″ apart, their common chord is 2·4″ in length, and the radius of the larger circle is 3·7″. Give a construction for finding the points of intersection of the two circles, and find the radius of the smaller circle.

Theorem 35. [Euclid III. 15.]

Of any two chords of a circle, that which is nearer to the centre is greater than one more remote.

Conversely, the greater of two chords is nearer to the centre than the less.



Let AB, CD be chords of a circle whose centre is O, and let OF, OG be perpendiculars on them from O.

It is required to prove that

(i) *if* OF *is less than* OG, *then* AB *is greater than* CD;

(ii) *if* AB *is greater than* CD, *then* OF *is less than* OG.

Join OA, OC.

Proof. Because OF is perp. to the chord AB,

∴ OF bisects AB;

∴ AF is half of AB.

Similarly CG is half of CD.

Now OA = OC,

∴ the sq. on OA = the sq. on OC.

But since the ∠OFA is a rt. angle,

∴ the sq. on OA = the sqq. on OF, FA.

Similarly the sq. on OC = the sqq. on OG, GC.

∴ the sqq. on OF, FA = the sqq. on OG, GC.

(i) Hence if OF is given less than OG;
the sq. on OF is less than the sq. on OG.
∴ the sq. on FA is greater than the sq. on GC;
∴ FA is greater than GC:
∴ AB is greater than CD.

(ii) But if AB is given greater than CD,
that is, if FA is greater than GC;
then the sq. on FA is greater than the sq. on GC.
∴ the sq. on OF is less than the sq. on OG;
∴ OF is less than OG. Q.E.D.

COROLLARY. *The greatest chord in a circle is a diameter.*

EXERCISES.

(Miscellaneous.)

1. Through a given point within a circle draw the least possible chord.

2. Draw a triangle ABC in which $a=3{\cdot}5''$, $b=1{\cdot}2''$, $c=3{\cdot}7''$. Through the ends of the side a draw a circle with its centre on the side c. Calculate and measure the radius.

3. Draw the circum-circle of a triangle whose sides are 2·6″, 2·8″, and 3·0″. Measure its radius.

4. AB is a fixed chord of a circle, and XY any other chord having its middle point Z on AB; what is the greatest, and what the least length that XY may have?

Shew that XY increases, as Z approaches the middle point of AB.

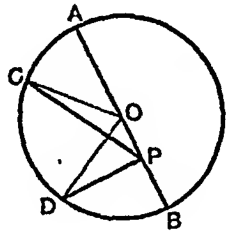
5. Shew on squared paper that a circle whose centre is at the origin, and whose radius is 3·0″, passes through the points (2·4″, 1·8″), (1·8″, 2·4″).

Find (i) the length of the chord joining these points, (ii) the co-ordinates of its middle point, (iii) its perpendicular distance from the origin.

*THEOREM 36. [Euclid III. 7.]

If from any internal point, not the centre, straight lines are drawn to the circumference of a circle, then the greatest is that which passes through the centre, and the least is the remaining part of that diameter.

And of any other two such lines the greater is that which subtends the greater angle at the centre.



Let ACDB be a circle, and from P any internal point, which is not the centre, let PA, PB, PC, PD be drawn to the O^{ce}, so that PA passes through the centre O, and PB is the remaining part of that diameter. Also let the ∠POC at the centre subtended by PC be greater than the ∠POD subtended by PD.

It is required to prove that of these st. lines

(i) PA *is the greatest,*

(ii) PB *is the least,*

(iii) PC *is greater than* PD.

Join OC, OD.

Proof. (i) In the △POC, the sides PO, OC are together greater than PC. *Theor.* 11.

But OC = OA, being radii;

∴ PO, OA are together greater than PC;

that is, PA is greater than PC.

Similarly PA may be shewn to be greater than any other st. line drawn from P to the O^{ce};

∴ PA is the greatest of all such lines.

(ii) In the $\triangle$ OPD, the sides OP, PD are together greater than OD.

But OD = OB, being radii;

$\therefore$ OP, PD are together greater than OB.

Take away the common part OP;

then PD is greater than PB.

Similarly any other st. line drawn from P to the O^{ce} may be shewn to be greater than PB;

$\therefore$ PB is the least of all such lines.

(iii) In the $\triangle^s$ POC, POD,

$$\text{because}\begin{cases}\text{PO is common,}\\ \text{OC}=\text{OD, being radii,}\\ \text{but the } \angle \text{POC is greater than the } \angle \text{POD;}\end{cases}$$

$\therefore$ PC is greater than PD. *Theor.* 19.

Q.E.D.

EXERCISES.

(*Miscellaneous.*)

1. *All circles which pass through a fixed point, and have their centres on a given straight line, pass also through a second fixed point.*

2. If two circles which intersect are cut by a straight line parallel to the common chord, shew that the parts of it intercepted between the circumferences are equal.

3. If two circles cut one another, any two parallel straight lines drawn through the points of intersection to cut the circles are equal.

4. If two circles cut one another, any two straight lines drawn through a point of section, making equal angles with the common chord, and terminated by the circumferences, are equal.

5. Two circles of diameters 74 and 40 inches respectively have a common chord 2 feet in length: find the distance between their centres.

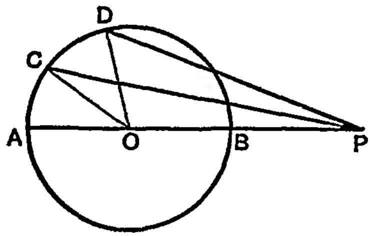
Draw the figure (1 cm. to represent 10″) and verify your result by measurement.

6. Draw two circles of radii 1·0″ and 1·7″, and with their centres 2·1″ apart. Find by calculation, and by measurement, the length of the common chord, and its distance from the two centres.

*Theorem 37. [Euclid III. 8.]

If from any external point straight lines are drawn to the circumference of a circle, the greatest is that which passes through the centre, and the least is that which when produced passes through the centre.

And of any other two such lines, the greater is that which subtends the greater angle at the centre.



Let ACDB be a circle, and from any external point P let the lines PBA, PC, PD be drawn to the O^{ce}, so that PBA passes through the centre O, and so that the ∠POC subtended by PC at the centre is greater than the ∠POD subtended by PD.

It is required to prove that of these st. lines

(i) PA *is the greatest,*
(ii) PB *is the least,*
(iii) PC *is greater than* PD.

Join OC, OD.

Proof. (i) In the △POC, the sides PO, OC are together greater than PC.

But OC = OA, being radii;

∴ PO, OA are together greater than PC;

that is, PA is greater than PC.

Similarly PA may be shewn to be greater than any other st. line drawn from P to the O^{ce};

that is, PA is the greatest of all such lines.

(ii) In the $\triangle$POD, the sides PD, DO are together greater than PO.

But OD = OB, being radii;

$\therefore$ the remainder PD is greater than the remainder PB.

Similarly any other st. line drawn from P to the O^{ce} may be shewn to be greater than PB;

that is, PB is the least of all such lines.

(iii) In the $\triangle^s$ POC, POD,

because $\begin{cases} \text{PO is common,} \\ \text{OC = OD, being radii;} \\ \text{but the } \angle \text{POC is greater than the } \angle \text{POD;} \end{cases}$

$\therefore$ PC is greater than PD. *Theor.* 19.

Q.E.D.

EXERCISES.

(Miscellaneous.)

1. Find the greatest and least straight lines which have one extremity on each of two given circles which do not intersect.

2. If from any point on the circumference of a circle straight lines are drawn to the circumference, the greatest is that which passes through the centre; and of any two such lines the greater is that which subtends the greater angle at the centre.

3. Of all straight lines drawn through a point of intersection of two circles, and terminated by the circumferences, the greatest is that which is parallel to the line of centres.

4. Draw on squared paper any two circles which have their centres on the x-axis, and cut at the point (8, −11). Find the coordinates of their other point of intersection.

5. Draw on squared paper two circles with centres at the points (15, 0) and (−6, 0) respectively, and cutting at the point (0, 8). Find the lengths of their radii, and the coordinates of their other point of intersection.

6. Draw an isosceles triangle OAB with an angle of 80° at its vertex O. With centre O and radius OA draw a circle, and on its circumference take any number of points P, Q, R, ... on the same side of AB as the centre. Measure the angles subtended by the chord AB at the points P, Q, R, ... Repeat the same exercise with any other given angle at O. What inference do you draw?

ON ANGLES IN SEGMENTS, AND ANGLES AT THE CENTRES AND CIRCUMFERENCES OF CIRCLES.

THEOREM 38. [Euclid III. 20.]

The angle at the centre of a circle is double of an angle at the circumference standing on the same arc.

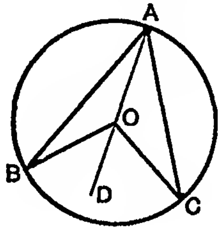


Fig. 1.

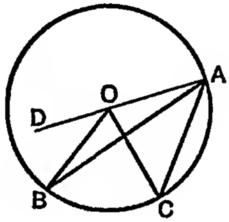


Fig. 2.

Let ABC be a circle, of which O is the centre; and let BOC be the angle at the centre, and BAC an angle at the $\odot^{ce}$, standing on the same arc BC.

It is required to prove that the $\angle$ BOC *is twice the* $\angle$ BAC.

Join AO, and produce it to D.

Proof. In the $\triangle$ OAB, because OB = OA,

$\therefore$ the $\angle$ OAB = the $\angle$ OBA.

$\therefore$ the sum of the $\angle^s$ OAB, OBA = twice the $\angle$ OAB.

But the ext. $\angle$ BOD = the sum of the $\angle^s$ OAB, OBA;

$\therefore$ the $\angle$ BOD = twice the $\angle$ OAB.

Similarly the $\angle$ DOC = twice the $\angle$ OAC.

$\therefore$, adding these results in Fig. 1, and taking the difference in Fig. 2, it follows in each case that

the $\angle$ BOC = twice the $\angle$ BAC. Q.E.D.

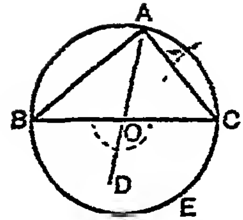


Fig. 3.

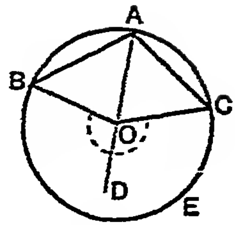


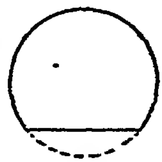
Fig. 4.

Obs. If the arc BEC, on which the angles stand, is a semi-circumference, as in Fig. 3, the ∠BOC at the centre is a *straight angle*; and if the arc BEC is greater than a semi-circumference, as in Fig. 4, the ∠BOC at the centre is *reflex*. But the proof for Fig. 1 applies without change to both these cases, shewing that whether the given arc is greater than, equal to, or less than a semi-circumference,

the ∠BOC = *twice the* ∠BAC, *on the same arc* BEC.

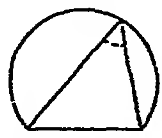
DEFINITIONS.

A segment of a circle is the figure bounded by a chord and one of the two arcs into which the chord divides the circumference.



NOTE. The chord of a segment is sometimes called its base.

An angle in a segment is one formed by two straight lines drawn from any point in the arc of the segment to the extremities of its chord.



We have seen in Theorem 32 that a circle may be drawn through *any three* points not in a straight line. But it is only under certain conditions that a circle can be drawn through more than three points.

DEFINITION. If four or more points are so placed that a circle may be drawn through them, they are said to be concyclic.

THEOREM 39. [Euclid III. 21.]

Angles in the same segment of a circle are equal.

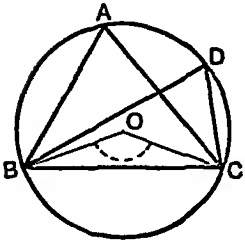


Fig. 1.

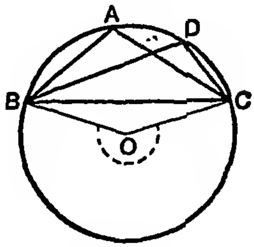


Fig. 2.

Let BAC, BDC be angles in the same segment BADC of a circle, whose centre is O.

It is required to prove that the ∠ BAC = *the* ∠ BDC.

Join BO, OC.

Proof. Because the ∠ BOC is at the centre, and the ∠ BAC at the O^{ce}, standing on the same arc BC,

∴ the ∠ BOC = twice the ∠ BAC. *Theor.* 38.

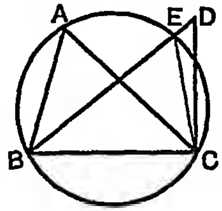
Similarly the ∠ BOC = twice the ∠ BDC.

∴ the ∠ BAC = the ∠ BDC. Q.E.D.

NOTE. The given segment may be greater than a semicircle as in Fig. 1, or less than a semicircle as in Fig. 2: in the latter case the angle BOC will be reflex. But by virtue of the extension of Theorem 38, given on the preceding page, the above proof applies equally to both figures.

Converse of Theorem 39.

Equal angles standing on the same base, and on the same side of it, have their vertices on an arc of a circle, of which the given base is the chord.



Let BAC, BDC be two equal angles standing on the same base BC, and on the same side of it.

It is required to prove that A *and* D *lie on an arc of a circle having* BC *as its chord.*

Let ABC be the circle which passes through the three points A, B, C; and suppose it cuts BD or BD produced at the point E.

Join EC.

Proof. Then the $\angle$BAC=the $\angle$BEC in the same segment.

But, by hypothesis, the $\angle$BAC=the $\angle$BDC;

$\therefore$ the $\angle$BEC=the $\angle$BDC;

which is impossible unless E coincides with D;

$\therefore$ the circle through B, A, C must pass through D.

Corollary. *The locus of the vertices of triangles drawn on the same side of a given base, and with equal vertical angles, is an arc of a circle.*

EXERCISES ON THEOREM 39.

1. In Fig. 1, if the angle BDC is 74°, find the number of degrees in each of the angles BAC, BOC, OBC.

2. In Fig. 2, let BD and CA intersect at X. If the angle DXC=40°, and the angle XCD=25°, find the number of degrees in the angle BAC and in the reflex angle BOC.

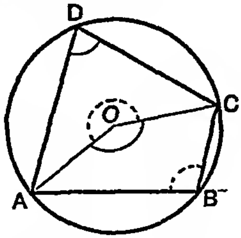
3. In Fig. 1, if the angles CBD, BCD are respectively 43° and 82°, find the number of degrees in the angles BAC, OBD, OCD.

4. Shew that in Fig. 2 the angle OBC is always less than the angle BAC by a right angle.

[For further Exercises on Theorem 39 see page 170.]

Theorem 40. [Euclid III. 22.]

The opposite angles of any quadrilateral inscribed in a circle are together equal to two right angles.



Let ABCD be a quadrilateral inscribed in the ⊙ABC.

It is required to prove that

(i) *the* ∠ˢ ADC, ABC *together* = *two rt. angles.*

(ii) *the* ∠ˢ BAD, BCD *together* = *two rt. angles.*

Suppose O is the centre of the circle.

Join OA, OC.

Proof. Since the ∠ADC at the Oᶜᵉ = half the ∠AOC at the centre, standing on the same arc ABC;
and the ∠ABC at the Oᶜᵉ = half the reflex ∠AOC at the centre, standing on the same arc ADC;
∴ the ∠ˢ ADC, ABC together = half the sum of the ∠AOC and the reflex ∠AOC.

But these angles make up four rt. angles.

∴ the ∠ˢ ADC, ABC together = two rt. angles.

Similarly the ∠ˢ BAD, BCD together = two rt. angles.

Q.E.D.

Note. The results of Theorems 39 and 40 should be carefully compared.

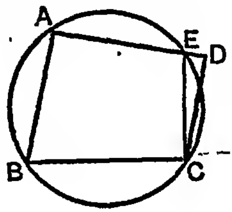
From Theorem 39 we learn that angles in the *same* segment are *equal.*

From Theorem 40 we learn that angles in *conjugate* segments are *supplementary.*

Definition. A quadrilateral is called cyclic when a circle can be drawn through its four vertices.

CONVERSE OF THEOREM 40.

If a pair of opposite angles of a quadrilateral are supplementary, its vertices are concyclic.



Let ABCD be a quadrilateral in which the opposite angles at B and D are supplementary.

It is required to prove that the points A, B, C, D *are concyclic.*

Let ABC be the circle which passes through the three points A, B, C; and suppose it cuts AD or AD produced in the point E.

Join EC.

Proof. Then since ABCE is a cyclic quadrilateral,

$\therefore$ the $\angle$AEC is the supplement of the $\angle$ABC.

But, by hypothesis, the $\angle$ADC is the supplement of the $\angle$ABC;

$\therefore$ the $\angle$AEC=the $\angle$ADC;

which is impossible unless E coincides with D.

$\therefore$ the circle which passes through A, B, C must pass through D:

that is, A, B, C, D are concyclic. Q.E.D.

EXERCISES ON THEOREM 40.

1. In a circle of 1·6″ radius inscribe a quadrilateral ABCD, making the angle ABC equal to 126°. Measure the remaining angles, and hence verify in this case that opposite angles are supplementary.

2. Prove Theorem 40 by the aid of Theorems 39 and 16, after first joining the opposite vertices of the quadrilateral.

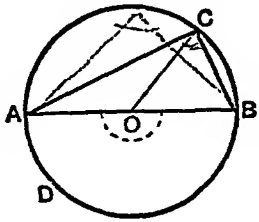
3. If a circle can be described about a parallelogram, the parallelogram must be rectangular.

4. ABC is an isosceles triangle, and XY is drawn parallel to the base BC cutting the sides in X and Y: shew that the four points B, C, X, Y lie on a circle. (/?)

5. *If one side of a cyclic quadrilateral is produced, the exterior angle is equal to the opposite interior angle of the quadrilateral.*

THEOREM 41. [Euclid III. 31.]

The angle in a semi-circle is a right angle.



Let ADB be a circle of which AB is a diameter and O the centre; and let C be any point on the semi-circumference ACB.

It is required to prove that the ∠ACB *is a rt. angle.*

1st Proof. The ∠ACB at the O^{ce} is half the *straight angle* AOB at the centre, standing on the same arc ADB;

and a *straight angle* = two rt. angles:

∴ the ∠ACB is a rt. angle. Q.E.D

2nd Proof. Join OC.

Then because OA = OC,

∴ the ∠OCA = the ∠OAC. *Theor.* 5.

And because OB = OC,

∴ the ∠OCB = the ∠OBC.

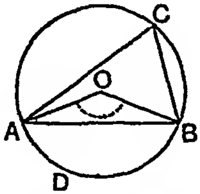
∴ the whole ∠ACB = the ∠OAC + the ∠OBC.

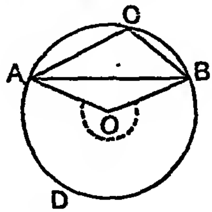
But the three angles of the △ACB together = two rt. angles;

∴ the ∠ACB = one-half of two rt. angles

= one rt. angle. Q.E.D

COROLLARY. *The angle in a segment greater than a semi-circle is acute; and the angle in a segment less than a semi-circle is obtuse.*





The $\angle$ ACB at the $\odot^{ce}$ is half the $\angle$ AOB at the centre, on the same arc ADB.

(i) If the segment ACB is greater than a semi-circle, then ADB is a *minor* arc;

∴ the $\angle$ AOB is *less* than two rt. angles;
∴ the $\angle$ ACB is *less* than one rt. angle.

(ii) If the segment ACB is less than a semi-circle, then ADB is a *major* arc;

∴ the $\angle$ AOB is *greater* than two rt. angles;
∴ the $\angle$ ACB is *greater* than one rt. angle.

EXERCISES ON THEOREM 41.

1. *A circle described on the hypotenuse of a right-angled triangle as diameter, passes through the opposite angular point.*

2. Two circles intersect at A and B; and through A two diameters AP, AQ are drawn, one in each circle: shew that the points P, B, Q are collinear.

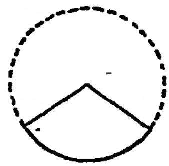
3. A circle is described on one of the equal sides of an isosceles triangle as diameter. Shew that it passes through the middle point of the base.

4. Circles described on any two sides of a triangle as diameters intersect on the third side, or the third side produced.

5. A straight rod of given length slides between two straight rulers placed at right angles to one another; find the locus of its middle point.

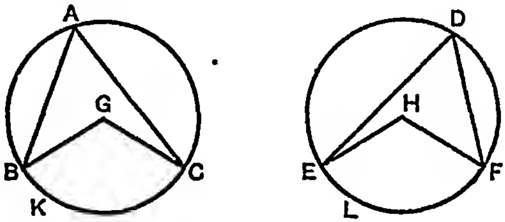
6. *Find the locus of the middle points of chords of a circle drawn through a fixed point. Distinguish between the cases when the given point is within, on, or without the circumference.*

DEFINITION. A sector of a circle is a figure bounded by two radii and the arc intercepted between them.



THEOREM 42. [Euclid III. 26.]

In equal circles, arcs which subtend equal angles, either at the centres or at the circumferences, are equal.



Let ABC, DEF be equal circles, and let the $\angle$ BGC = the $\angle$ EHF at the centres; and consequently

the $\angle$ BAC = the $\angle$ EDF at the $\odot^{ces}$. *Theor.* 38.

It is required to prove that the arc BKC = *the arc* ELF.

Proof. Apply the $\odot$ABC to the $\odot$DEF, so that the centre G falls on the centre H, and GB falls along HE.

Then because the $\angle$ BGC = the $\angle$ EHF,

$\therefore$ GC will fall along HF.

And because the circles have equal radii, B will fall on E, and C on F, and the circumferences of the circles will coincide entirely.

$\therefore$ the arc BKC must coincide with the arc ELF;

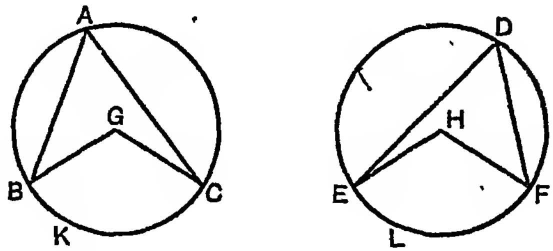
$\therefore$ the arc BKC = the arc ELF. Q.E.D.

COROLLARY. *In equal circles sectors which have equal angles are equal.*

Obs. It is clear that any theorem relating to arcs, angles, and chords in *equal* circles must also be true in *the same* circle.

THEOREM 43. [Euclid III. 27.]

In equal circles angles, either at the centres or at the circumferences, which stand on equal arcs are equal.



Let ABC, DEF be equal circles;
and let the arc BKC = the arc ELF.

It is required to prove that
the ∠ BGC = *the* ∠ EHF *at the centres;*
also the ∠ BAC = *the* ∠ EDF *at the* ⊙ces.

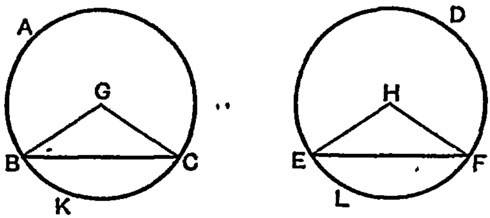
Proof. Apply the ⊙ ABC to the ⊙ DEF, so that the centre G falls on the centre H, and GB falls along HE.

Then because the circles have equal radii,
∴ B falls on E, and the two ⊙ces coincide entirely.
And, by hypothesis, the arc BKC = the arc ELF.
∴ C falls on F, and consequently GC on HF;
∴ the ∠ BGC = the ∠ EHF.

And since the ∠ BAC at the ⊙ce = half the ∠ BGC at the centre;
and likewise the ∠ EDF = half the ∠ EHF;
∴ the ∠ BAC = the ∠ EDF. Q.E.D.

THEOREM 44. [Euclid III. 28.]

In equal circles, arcs which are cut off by equal chords are equal, the major arc equal to the major arc, and the minor to the minor.



Let ABC, DEF be equal circles whose centres are G and H;
and let the chord BC = the chord EF.

It is required to prove that

the major arc BAC = *the major arc* EDF,

and *the minor arc* BKC = *the minor arc* ELF.

Join BG, GC, EH, HF.

Proof. In the $\triangle^s$ BGC, EHF,

because $\begin{cases} \text{BG = EH, being radii of equal circles,} \\ \text{GC = HF, for the same reason,} \\ \text{and BC = EF, by hypothesis;} \end{cases}$

$\therefore$ the $\angle$ BGC = the $\angle$ EHF; *Theor.* 7.

$\therefore$ the arc BKC = the arc ELF; *Theor.* 42.

and these are the minor arcs.

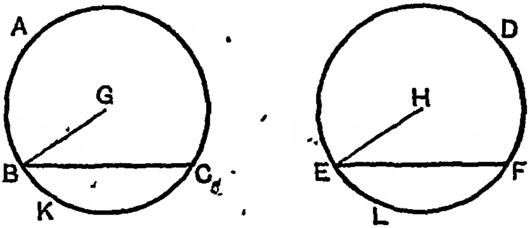
But the whole $\odot^{ce}$ ABKC = the whole $\odot^{ce}$ DELF;

$\therefore$ the remaining arc BAC = the remaining arc EDF:

and these are the major arcs. Q.E.D.

THEOREM 45. [Euclid III. 29.]

In equal circles chords which cut off equal arcs are equal.



Let ABC, DEF be equal circles whose centres are G and H; and let the arc BKC = the arc ELF.

It is required to prove that the chord BC = *the chord* EF.

Join BG, EH.

Proof. Apply the ⊙ABC to the ⊙DEF, so that G falls on H and GB along HE.

Then because the circles have equal radii,

∴ B falls on E, and the ⊙ces coincide entirely.

And because the arc BKC = the arc ELF,

∴ C falls on F.

∴ the chord BC coincides with the chord EF;

∴ the chord BC = the chord EF. Q.E.D.

EXERCISES ON ANGLES IN A CIRCLE.

1. P is any point on the arc of a segment of which AB is the chord. Shew that the sum of the angles PAB, PBA is constant.

2. PQ and RS are two chords of a circle intersecting at X: prove that the triangles PXS, RXQ are equiangular to one another.

3. Two circles intersect at A and B; and through A any straight line PAQ is drawn terminated by the circumferences: shew that PQ subtends a constant angle at B.

4. Two circles intersect at A and B; and through A any two straight lines PAQ, XAY are drawn terminated by the circumferences; shew that the arcs PX, QY subtend equal angles at B.

5. P is any point on the arc of a segment whose chord is AB: and the angles PAB, PBA are bisected by straight lines which intersect at O. Find the locus of the point O.

6. *If two chords intersect within a circle, they form an angle equal to that at the centre, subtended by half the sum of the arcs they cut off.*

7. *If two chords intersect without a circle, they form an angle equal to that at the centre subtended by half the difference of the arcs they cut off.*

8. The sum of the arcs cut off by two chords of a circle at right angles to one another is equal to the semi-circumference.

9. *If* AB *is a fixed chord of a circle and* P *any point on one of the arcs cut off by it, then the bisector of the angle* APB *cuts the conjugate arc in the same point for all positions of* P.

10. AB, AC are any two chords of a circle; and P, Q are the middle points of the minor arcs cut off by them; if PQ is joined, cutting AB in X and AC in Y, shew that AX=AY.

11. A triangle ABC is inscribed in a circle, and the bisectors of the angles meet the circumference at X, Y, Z. Shew that the angles of the triangle XYZ are respectively

$$90^\circ - \frac{A}{2}, \quad 90^\circ - \frac{B}{2}, \quad 90^\circ - \frac{C}{2}.$$

12. Two circles intersect at A and B; and through these points lines are drawn from any point P on the circumference of one of the circles: shew that when produced they intercept on the other circumference an arc which is constant for all positions of P.

13. The straight lines which join the extremities of parallel chords in a circle (i) towards the same parts, (ii) towards opposite parts, are equal.

14. Through A, a point of intersection of two equal circles, two straight lines PAQ, XAY are drawn: shew that the chord PX is equal to the chord QY.

15. Through the points of intersection of two circles two parallel straight lines are drawn terminated by the circumferences: shew that the straight lines which join their extremities towards the same parts are equal.

16. Two equal circles intersect at A and B; and through A any straight line PAQ is drawn terminated by the circumferences: shew that BP=BQ.

17. ABC is an isosceles triangle inscribed in a circle, and the bisectors of the base angles meet the circumference at X and Y. Shew that the figure BXAYC must have four of its sides equal.

What relation must subsist among the angles of the triangle ABC, in order that the figure BXAYC may be equilateral?

18. ABCD is a cyclic quadrilateral, and the opposite sides AB, DC are produced to meet at P, and CB, DA to meet at Q: if the circles circumscribed about the triangles PBC, QAB intersect at R, shew that the points P, R, Q are collinear.

19. P, Q, R *are the middle points of the sides of a triangle, and* X *is the foot of the perpendicular let fall from one vertex on the opposite side: shew that the four points* P, Q, R, X *are concyclic.*

[See page 64, Ex. 2: also Prob. 10, p. 83.]

20. *Use the preceding exercise to shew that the middle points of the sides of a triangle and the feet of the perpendiculars let fall from the vertices on the opposite sides, are concyclic.*

21. If a series of triangles are drawn standing on a fixed base, and having a given vertical angle, shew that the bisectors of the vertical angles all pass through a fixed point.

22. ABC is a triangle inscribed in a circle, and E the middle point of the arc subtended by BC on the side remote from A: if through E a diameter ED is drawn, shew that the angle DEA is half the difference of the angles at B and C.

TANGENCY.

Definitions and First Principles.

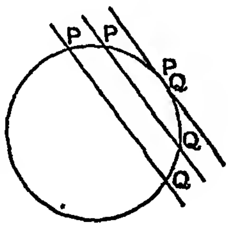
1. A secant of a circle is a straight line of indefinite length which cuts the circumference at two points.

2. If a secant moves in such a way that the two points in which it cuts the circle continually approach one another, then in the ultimate position when these two points become one, the secant becomes a tangent to the circle, and is said to touch it at the point at which the two intersections coincide. This point is called the **point of contact.**

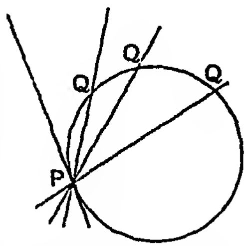
For instance:

(i) Let a secant cut the circle at the points P and Q, and suppose it to recede from the centre, moving always parallel to its original position; then the two points P and Q will clearly approach one another and finally coincide.

In the ultimate position when P and Q become one point, the straight line becomes a tangent to the circle at that point.



(ii) Let a secant cut the circle at the points P and Q, and suppose it to be turned about the point P so that while P remains fixed, Q moves on the circumference nearer and nearer to P. Then the line PQ in its ultimate position, when Q coincides with P, is a tangent at the point P.



Since a secant can cut a circle at *two* points only, it is clear that a tangent can have only one point in common with the circumference, namely the point of contact, at which two points of section coincide. Hence we may define a tangent as follows:

3. A tangent to a circle is a straight line which meets the circumference at one point only; and though produced indefinitely does not cut the circumference.

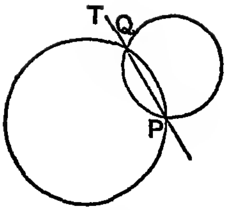


Fig. 1.

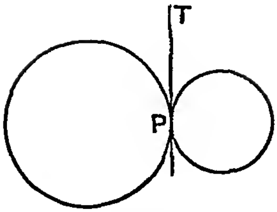


Fig. 2.

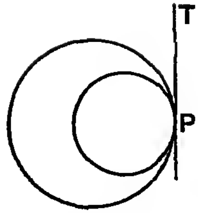


Fig. 3.

4. Let two circles intersect (as in Fig. 1) in the points P and Q, and let one of the circles turn about the point P, which remains fixed, in such a way that Q continually approaches P. Then in the ultimate position, when Q coincides with P (as in Figs. 2 and 3), the circles are said to touch one another at P.

Since two circles cannot intersect in more than *two* points, two circles which touch one another cannot have more than *one* point in common, namely the point of contact at which the two points of section coincide. Hence circles are said to touch one another when they meet, but do not cut one another.

NOTE. When each of the circles which meet is *outside the other*, as in Fig. 2, they are said to touch one another **externally**, or to have **external contact**: when one of the circles is *within the other*, as in Fig. 3, the first is said to touch the other **internally**, or to have **internal contact** with it.

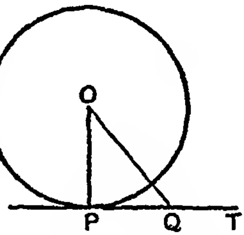
INFERENCE FROM DEFINITIONS 2 AND 4.

If in Fig. 1, TQP is a common chord of two circles one of which is made to turn about P, then when Q is brought into coincidence with P, the line TP passes through two coincident points on each circle, as in Figs. 2 and 3, and therefore becomes a tangent to each circle. Hence

Two circles which touch one another have a common tangent at their point of contact.

THEOREM 46.

The tangent at any point of a circle is perpendicular to the radius drawn to the point of contact.



Let PT be a tangent at the point P to a circle whose centre is O.

It is required to prove that PT *is perpendicular to the radius* OP.

Proof. Take *any* point Q in PT, and join OQ.

Then since PT is a tangent, every point in it except P is outside the circle.

∴ OQ is greater than the radius OP.

And this is true for every point Q in PT;

∴ OP is the shortest distance from O to PT.

Hence OP is perp. to PT. *Theor.* 12, *Cor.* 1.

Q.E.D.

COROLLARY 1. Since there can be only one perpendicular to OP at the point P, it follows that *one and only one tangent can be drawn to a circle at a given point on the circumference.*

COROLLARY 2. Since there can be only one perpendicular to PT at the point P, it follows that *the perpendicular to a tangent at its point of contact passes through the centre.*

COROLLARY 3. Since there can be only one perpendicular from O to the line PT, it follows that *the radius drawn perpendicular to the tangent passes through the point of contact.*

THEOREM 46. [By the Method of Limits.]

The tangent at any point of a circle is perpendicular to the radius drawn to the point of contact.

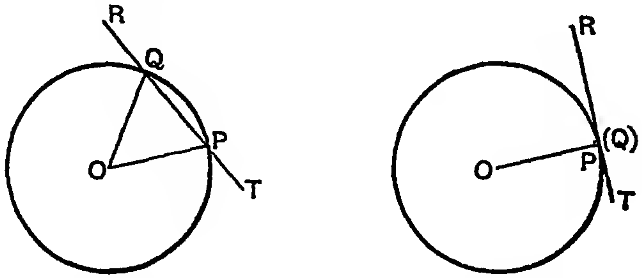


Fig. 1. Fig. 2.

Let P be a point on a circle whose centre is O.

It is required to prove that the tangent at P *is perpendicular to the radius* OP.

Let RQPT (Fig. 1) be a secant cutting the circle at Q and P.

Join OQ, OP.

Proof. Because OP = OQ,

∴ the ∠OQP = the ∠OPQ;

∴ the supplements of these angles are equal;

that is, the ∠OQR = the ∠OPT,

and this is true however near Q is to P.

Now let the secant QP be turned about the point P so that Q continually approaches and finally coincides with P; then in the ultimate position,

(i) the secant RT becomes the tangent at P, } Fig. 2,
(ii) OQ coincides with OP;

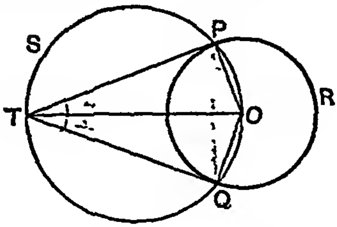
and therefore the equal ∠s OQR, OPT become adjacent,

∴ OP is perp. to RT. Q.E.D.

NOTE. The method of proof employed here is known as the **Method of Limits.**

THEOREM 47.

Two tangents can be drawn to a circle from an external point.



Let PQR be a circle whose centre is O, and let T be an external point.

It is required to prove that there can be two tangents drawn to the circle from T.

Join OT, and let TSO be the circle on OT as diameter.

This circle will cut the ⊙PQR in two points, since T is without, and O is within, the ⊙PQR. Let P and Q be these points.

Join TP, TQ; OP, OQ.

Proof. Now each of the ∠^{s} TPO, TQO, being in a semicircle, is a rt. angle;

∴ TP, TQ are perp. to the radii OP, OQ respectively.

∴ TP, TQ are tangents at P and Q. *Theor.* 46.

Q.E.D.

COROLLARY. *The two tangents to a circle from an external point are equal, and subtend equal angles at the centre.*

For in the △^{s} TPO, TQO,

because $\begin{cases} \text{the } \angle^s \text{ TPO, TQO are } \textit{right angles}, \\ \text{the hypotenuse TO is common,} \\ \text{and OP} = \text{OQ, being radii;} \end{cases}$

∴ TP = TQ,

and the ∠TOP = the ∠TOQ. *Theor.* 18.

EXERCISES ON THE TANGENT.

(Numerical and Graphical.)

1. Draw two concentric circles with radii 5·0 cm. and 3·0 cm. Draw a series of chords of the former to touch the latter. Calculate and measure their lengths, and account for their being equal.

2. In a circle of radius 1·0″ draw a number of chords each 1·6″ in length. Shew that they all touch a concentric circle, and find its radius.

3. The diameters of two concentric circles are respectively 10·0 cm. and 5·0 cm.: find to the nearest millimetre the length of any chord of the outer circle which touches the inner, and check your work by measurement.

4. In the figure of Theorem 47, if OP=5″, TO=13″, find the length of the tangents from T. Draw the figure (scale 2 cm. to 5″), and measure to the nearest degree the angles subtended at O by the tangents.

5. The tangents from T to a circle whose radius is 0·7″ are each 2·4″ in length. Find the distance of T from the centre of the circle. Draw the figure and check your result graphically.

(Theoretical.)

6. *The centre of any circle which touches two intersecting straight lines must lie on the bisector of the angle between them.*

7. AB and AC are two tangents to a circle whose centre is O; shew that AO bisects the chord of contact BC at right angles.

8. If PQ is joined in the figure of Theorem 47, shew that the angle PTQ is double the angle OPQ.

9. Two parallel tangents to a circle intercept on any third tangent a segment which subtends a right angle at the centre.

10. The diameter of a circle bisects all chords which are parallel to the tangent at either extremity.

11. *Find the locus of the centres of all circles which touch a given straight line at a given point.*

12. Find the locus of the centres of all circles which touch each of two parallel straight lines.

13. *Find the locus of the centres of all circles which touch each of two intersecting straight lines of unlimited length.*

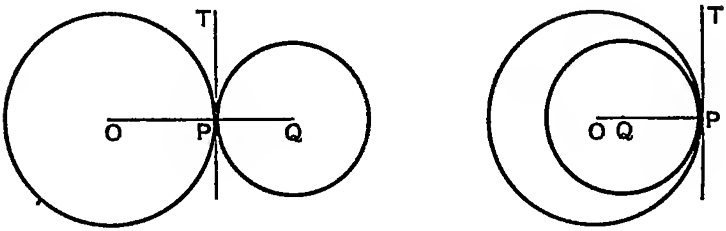
14. *In any quadrilateral circumscribed about a circle, the sum of one pair of opposite sides is equal to the sum of the other pair.*

State and prove the converse theorem.

15. If a quadrilateral is described about a circle, the angles subtended at the centre by any two opposite sides are supplementary.

THEOREM 48.

If two circles touch one another, the centres and the point of contact are in one straight line.



Let two circles whose centres are O and Q touch at the point P.

It is required to prove that O, P, *and* Q *are in one straight line.*

Join OP, QP.

Proof. Since the given circles touch at P, they have a common tangent at that point. *Page* 173.

Suppose PT to touch both circles at P.

Then since OP and QP are radii drawn to the point of contact,

∴ OP and QP are both perp. to PT;

∴ OP and QP are in one st. line. *Theor.*

That is, the points O, P, and Q are in one st. line Q.E.D.

COROLLARIES. (i) *If two circles touch externally the distan between their centres is equal to the sum of their radii.*

(ii) *If two circles touch internally the distance between th centres is equal to the difference of their radii.*

EXERCISES ON THE CONTACT OF CIRCLES.

(Numerical and Graphical.)

1. From centres 2·6″ apart draw two circles with radii 1·7″ and 0·9″ respectively. Why and where do these circles touch one another?

If circles of the above radii are drawn from centres 0·8″ apart, prove that they touch. How and why does the contact differ from that in the former case?

2. Draw a triangle ABC in which $a=8$ cm., $b=7$ cm., and $c=6$ cm. From A, B, and C as centres draw circles of radii 2·5 cm., 3·5 cm., and 4·5 cm. respectively; and shew that these circles touch in pairs.

3. In the triangle ABC, right-angled at C, $a=8$ cm. and $b=6$ cm.; and from centre A with radius 7 cm. a circle is drawn. What must be the radius of a circle drawn from centre B to touch the first circle?

4. A and B are the centres of two fixed circles which touch internally. If P is the centre of any circle which touches the larger circle internally and the smaller externally, prove that AP + BP is constant.

If the fixed circles have radii 5·0 cm. and 3·0 cm. respectively, verify the general result by taking different positions for P.

5. AB is a line 4″ in length, and C is its middle point. On AB, AC, CB semicircles are described. Shew that if a circle is inscribed in the space enclosed by the three semicircles its radius must be $\frac{2}{3}$″.

(Theoretical.)

6. *A straight line is drawn through the point of contact of two circles whose centres are* A *and* B, *cutting the circumferences at* P *and* Q *respectively; shew that the radii* AP *and* BQ *are parallel.*

7. Two circles touch externally, and through the point of contact a straight line is drawn terminated by the circumferences; show that the tangents at its extremities are parallel.

8. Find the locus of the centres of all circles

(i) which touch a given circle at a given point;

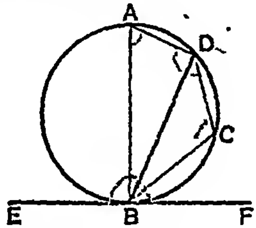
(ii) which are of given radius and touch a given circle.

9. From a given point as centre describe a circle to touch a given circle. How many solutions will there be?

10. Describe a circle of radius a to touch a given circle of radius b at a given point. How many solutions will there be?

THEOREM 49. [Euclid III. 32.]

The angles made by a tangent to a circle with a chord drawn from the point of contact are respectively equal to the angles in the alternate segments of the circle.



Let EF touch the ⊙ABC at B, and let BD be a chord drawn from B, the point of contact.

It is required to prove that

(i) *the* ∠FBD = *the angle in the alternate segment* BAD;

(ii) *the* ∠EBD = *the angle in the alternate segment* BCD.

Let BA be the diameter through B, and C any point in the arc of the segment which does not contain A.

Join AD, DC, CB.

Proof. Because the ∠ADB in a semi-circle is a rt. angle,

∴ the ∠s DBA, BAD together = a rt. angle.

But since EBF is a tangent, and BA a diameter,

∴ the ∠FBA is a rt. angle.

∴ the ∠FBA = the ∠s DBA, BAD together.

Take away the common ∠DBA,

then the ∠FBD = the ∠BAD, which is in the alternate segment.

Again because ABCD is a cyclic quadrilateral,

∴ the ∠BCD = the supplement of the ∠BAD

= the supplement of the ∠FBD

= the ∠EBD;

the ∠EBD = the ∠BCD, which is in the alternate segment.

Q.E.D.

EXERCISES ON THEOREM 49.

1. In the figure of Theorem 49, if the $\angle FBD = 72°$, write down the values of the $\angle^s BAD, BCD, EBD$.

2. Use this theorem to shew that tangents to a circle from an external point are equal.

3. Through A, the point of contact of two circles, chords APQ, AXY are drawn: shew that PX and QY are parallel.

Prove this (i) for internal, (ii) for external contact.

4. AB is the common chord of two circles, one of which passes through O, the centre of the other: prove that OA bisects the angle between the common chord and the tangent to the first circle at A.

5. Two circles intersect at A and B; and through P, any point on one of them, straight lines PAC, PBD are drawn to cut the other at C and D: shew that CD is parallel to the tangent at P.

6. If from the point of contact of a tangent to a circle a chord is drawn, the perpendiculars dropped on the tangent and chord from the middle point of either arc cut off by the chord are equal.

EXERCISES ON THE METHOD OF LIMITS.

1. *Prove Theorem 49 by the Method of Limits.*

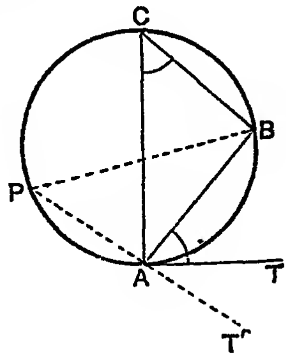
[Let ACB be a segment of a circle of which AB is the chord; and let PAT′ be any *secant* through A. Join PB.

Then the $\angle BCA =$ the $\angle BPA$; *Theor.* 39.

and this is true *however near* P *approaches to* A.

If P moves up to coincidence with A, then the secant PAT′ becomes the tangent AT, and the $\angle BPA$ becomes the $\angle BAT$.

$\therefore$, ultimately, the $\angle BAT =$ the $\angle BCA$, in the alt. segment.]



2. From Theorem 31, prove by the Method of Limits that

The straight line drawn perpendicular to the diameter of a circle at its extremity is a tangent.

3. Deduce Theorem 48 from the property that *the line of centres bisects a common chord at right angles.*

4. Deduce Theorem 49 from Ex. 5, page 163.

5. Deduce Theorem 46 from Theorem 41.

PROBLEMS.

GEOMETRICAL ANALYSIS.

Hitherto the Propositions of this text-book have been arranged Synthetically, that is to say, by *building up known results* in order to obtain a *new* result.

But this arrangement, though convincing as an argument, in most cases affords little clue as to the way in which the construction or proof *was discovered.* We therefore draw the student's attention to the following hints.

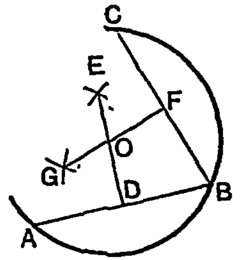
In attempting to solve a problem begin by *assuming* the required result; then by working backwards, trace the consequences of the assumption, and try to ascertain its dependence on some condition or known theorem which suggests the necessary construction. If this attempt is successful, the steps of the argument may in general be re-arranged in reverse order, and the construction and proof presented in a synthetic form.

This unravelling of the conditions of a proposition in order to trace it back to some earlier principle on which it depends, is called **geometrical analysis**: it is the natural way of attacking the harder types of exercises, and it is especially useful in solving problems.

Although the above directions do not amount to a *method,* they often furnish a very effective mode of *searching for a suggestion.* The approach by analysis will be illustrated in some of the following problems. [See Problems 23, 28, 29.]

PROBLEM 20.

Given a circle, or an arc of a circle, to find its centre.



Let ABC be an arc of a circle whose centre is to be found.

Construction. Take two chords AB, BC, and bisect them at right angles by the lines DE, FG, meeting at O. *Prob.* 2.

Then O is the required centre.

Proof. Every point in DE is equidistant from A and B. *Prob.* 14.

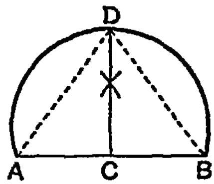
And every point in FG is equidistant from B and C.

∴ O is equidistant from A, B, and C.

∴ O is the centre of the circle ABC. *Theor.* 33.

PROBLEM 21.

To bisect a given arc.



Let ADB be the given arc to be bisected.

Construction. Join AB, and bisect it at right angles by CD meeting the arc at D. *Prob.* 2.

Then the arc is bisected at D.

Proof. Join DA, DB.

Then every point on CD is equidistant from A and B; *Prob.* 14.

∴ DA = DB;

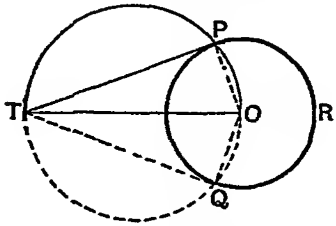
∴ the ∠ DBA = the ∠ DAB; *Theorem* 6.

∴ the arcs, which subtend these angles at the O^ce^, are equal;

that is, the arc DA = the arc DB.

PROBLEM 22.

To draw a tangent to a circle from a given external point.



Let PQR be the given circle, with its centre at O; and let T be the point from which a tangent is to be drawn.

Construction. Join TO, and on it describe a semi-circle TPO to cut the circle at P.

Join TP.

Then TP is the required tangent.

Proof. Join OP.

Then since the ∠TPO, being in a semi-circle, is a rt. angle,

∴ TP is at right angles to the radius OP.

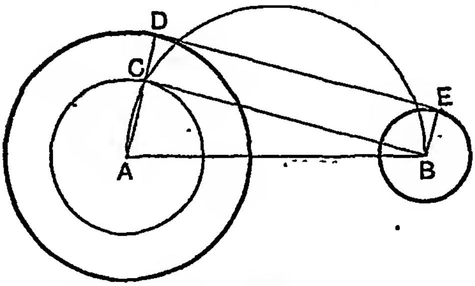
∴ TP is a tangent at P. *Theor.* 46.

Since the semi-circle may be described on either side of TO, a second tangent TQ can be drawn from T, as shown in the figure.

NOTE. *Suppose the point* T *to approach the given circle, then the* angle PTQ gradually increases. When T reaches the circumference, the angle PTQ becomes a *straight angle*, and the two tangents coincide. When T enters the circle, no tangent can be drawn. [See *Obs.* p. 94.]

PROBLEM 23.

To draw a common tangent to two circles.



Let A be the centre of the greater circle, and a its radius; and let B be the centre of the smaller circle, and b its radius.

Analysis. Suppose DE to touch the circles at D and E. Then the radii AD, BE are both perp. to DE, and therefore par[1] to one another.

Now if BC were drawn par[1] to DE, then the fig. DB would be a rectangle, so that CD = BE = b.

And if AD, BE are *on the same side of* AB,

then AC = $a - b$, and the ∠ACB is a rt. angle.

These hints enable us to draw BC *first*, and thus lead to the following construction.

Construction. With centre A, and radius equal to the *difference* of the radii of the given circles, describe a circle, and draw BC to touch it.

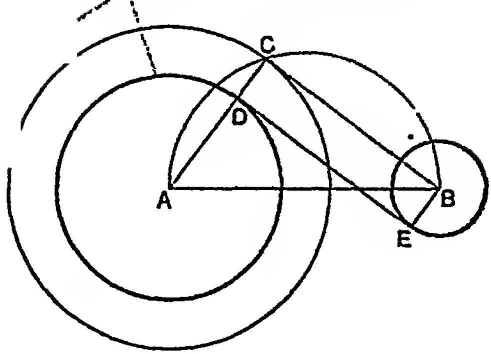
Join AC, and produce it to meet the circle (A) at D.

Through B draw the radius BE par[1] to AD and in the *same sense.* Join DE.

Then DE is a common tangent to the given circles.

Obs. Since two tangents, such as BC, can in general be drawn from B to the circle of construction, this method will furnish two common tangents to the given circles. These are called the direct common tangents.

PROBLEM 23. (*Continued.*)



Again, if the circles are *external to one another* two more common tangents may be drawn.

Analysis. In this case we may suppose DE to touch the circles at D and E so that the radii AD, BE fall *on opposite sides of* AB.

Then BC, drawn par[l] to the supposed common tangent DE, would meet AD *produced* at C; and we should now have

AC = AD + DC = $a + b$; and, as before, the $\angle$ ACB is a rt. angle.

Hence the following construction.

Construction. With centre A, and radius equal to the *sum* of the radii of the given circles, describe a circle, and draw BC to touch it.

Then proceed as in the first case, but draw BE in the sense *opposite* to AD.

Obs. As before, two tangents may be drawn from B to the circle of construction; hence two common tangents may be thus drawn to the given circles These are called the transverse common tangents.

[We leave as an exercise to the student the arrangement of the proof in synthetic form.]

EXERCISES ON COMMON TANGENTS.

(Numerical and Graphical.)

1. How many common tangents can be drawn in each of the following cases?

(i) when the given circles intersect;

(ii) when they have external contact;

(iii) when they have internal contact.

Illustrate your answer by drawing two circles of radii 1·4″ and 1·0″ respectively,

(i) with 1·0″ between the centres;

(ii) with 2·4″ between the centres;

(iii) with 0·4″ between the centres;

(iv) with 3·0″ between the centres.

Draw the common tangents in each case, and note where the general construction fails, or is modified.

2. Draw two circles with radii 2·0″ and 0·8″, placing their centres 2·0″ apart. Draw the common tangents, and find their lengths between the points of contact, both by calculation and by measurement.

3. Draw all the common tangents to two circles whose centres are 1·8″ apart and whose radii are 0·6″ and 1·2″ respectively. Calculate and measure the length of the direct common tangents.

4. Two circles of radii 1·7″ and 1·0″ have their centres 2·1″ apart. Draw their common tangents and find their lengths. Also find the length of the common chord. Produce the common chord and shew by measurement that it bisects the common tangents.

5. Draw two circles with radii 1·6″ and 0·8″ and with their centres 3·0″ apart. Draw all their common tangents.

6. Draw the direct common tangents to two equal circles.

(Theoretical.)

7. If the two direct, or the two transverse, common tangents are drawn to two circles, the parts of the tangents intercepted between the points of contact are equal.

8. If four common tangents are drawn to two circles external to one another, shew that the two direct, and also the two transverse, tangents intersect on the line of centres.

9. Two given circles have external contact at A, and a direct common tangent is drawn to touch them at P and Q: shew that PQ subtends a right angle at the point A.

ON THE CONSTRUCTION OF CIRCLES.

In order to draw a circle we must know (i) the position of the centre, (ii) the length of the radius.

(i) To find the position of the centre, two conditions are needed, each giving a locus on which the centre must lie; so that the one or more points in which the two loci intersect are possible positions of the required centre, as explained on page 93.

(ii) The position of the centre being thus fixed, the radius is determined if we know (or can find) any point on the circumference.

Hence in order to draw a circle *three* independent data are required.

For example, we may draw a circle if we are given

(i) *three* points on the circumference;

or (ii) *three* tangent lines;

or (iii) one point on the circumference, one tangent, and its point of contact.

It will however often happen that more than one circle can be drawn satisfying three given conditions.

Before attempting the constructions of the next Exercise the student should make himself familiar with the following loci.

(i) *The locus of the centres of circles which pass through two given points.*

(ii) *The locus of the centres of circles which touch a given straight line at a given point.*

(iii) *The locus of the centres of circles which touch a given circle at a given point.*

(iv) *The locus of the centres of circles which touch a given straight line, and have a given radius.*

(v) *The locus of the centres of circles which touch a given circle, and have a given radius.*

(vi) *The locus of the centres of circles which touch two given straight lines.*

EXERCISES.

1. Draw a circle to pass through three given points.

2. If a circle touches a given line PQ at a point A, on what line must its centre lie?

If a circle passes through two given points A and B, on what line must its centre lie?

Hence draw a circle to touch a straight line PQ at the point A, and to pass through another given point B.

3. If a circle touches a given circle whose centre is C at the point A, on what line must its centre lie?

Draw a circle to touch the given circle (C) at the point A, and to pass through a given point B.

4. A point P is 4·5 cm. distant from a straight line AB. Draw two circles of radius 3·2 cm. to pass through P and to touch AB.

5. Given two circles of radius 3·0 cm. and 2·0 cm. respectively, their centres being 6·0 cm. apart; draw a circle of radius 3·5 cm. to touch each of the given circles externally.

How many solutions will there be? What is the radius of the smallest circle that touches each of the given circles externally?

6. If a circle touches two straight lines OA, OB, on what line must its centre lie?

Draw OA, OB, making an angle of 76°, and describe a circle of radius 1·2″ to touch both lines.

7. Given a circle of radius 3·5 cm., with its centre 5·0 cm. from a given straight line AB; draw two circles of radius 2·5 cm. to touch the given circle and the line AB.

8. Devise a construction for drawing a circle to touch each of two parallel straight lines and a transversal.

Shew that two such circles can be drawn, and that they are equal.

9. Describe a circle to touch a given circle, and also to touch a given straight line at a given point. [See page 311.]

10. Describe a circle to touch a given straight line, and to touch a given circle at a given point.

circumference

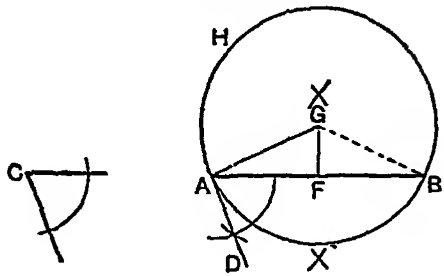
11. Shew how to draw a circle to touch each of three given straight lines of which no two are parallel.

How many such circles can be drawn?

[Further Examples on the Construction of Circles will be found on pp. 246, 311.]

PROBLEM 24.

On a given straight line to describe a segment of a circle which shall contain an angle equal to a given angle.



Let AB be the given st line, and C the given angle.

It is required to describe on AB *a segment of a circ' containing an angle equal to* C.

Construction. At A in BA, make the ∠ BAD equal to the ∠ C.

From A draw AG perp. to AD.

Bisect AB at rt. angles by FG, meeting AG in G. *Prob.* 2.

Proof. Join GB.

Now every point in FG is equidistant from A and B; *Prob.* 14.

∴ GA = GB.

With centre G, and radius GA, draw a circle, which must pass through B, and touch AD at A. *Theor.* 46.

Then the segment AHB, alternate to the ∠ BAD, contains an angle equal to C. *Theor.* 49.

NOTE. In the particular case when the given angle is a rt. angle, the segment required will be the semi-circle on AB as diameter. [Theorem 41.]

Corollary. *To cut off from a given circle a segment containing a given angle, it is enough to draw a tangent to the circle, and from the point of contact to draw a chord making with the tangent an angle equal to the given angle.*

It was proved on page 161 that

The locus of the vertices of triangles which stand on the same base and have a given vertical angle, is the arc of the segment standing on this base, and containing an angle equal to the given angle.

The following Problems are derived from this result by the Method of Intersection of Loci [page 93].

EXERCISES.

1. *Describe a triangle on a given base having a given vertical angle and having its vertex on a given straight line.*

2. *Construct a triangle having given the base, the vertical angle, and*
 (i) *one other side.*
 (ii) *the altitude.*
 (iii) *the length of the median which bisects the base.*
 (iv) *the foot of the perpendicular from the vertex to the base.*

3. *Construct a triangle having given the base, the vertical angle, and the point at which the base is cut by the bisector of the vertical angle.*

[Let AB be the base, X the given point in it, and K the given angle. On AB describe a segment of a circle containing an angle equal to K; complete the O^ce by drawing the arc APB. Bisect the arc APB at P: join PX, and produce it to meet the O^ce at C. Then ABC is the required triangle.]

4. *Construct a triangle having given the base, the vertical angle, and the sum of the remaining sides.*

[Let AB be the given base, K the given angle, and H a line equal to the sum of the sides. On AB describe a segment containing an angle equal to K, also another segment containing an angle equal to half the ∠K. With centre A, and radius H, describe a circle cutting the arc of the latter segment at X and Y. Join AX (or AY) cutting the arc of the first segment at C. Then ABC is the required triangle.]

5. *Construct a triangle having given the base, the vertical angle, and the difference of the remaining sides.*

CIRCLES IN RELATION TO RECTILINEAL FIGURES.

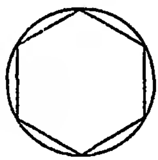
DEFINITIONS.

1. A **Polygon** is a rectilineal figure bounded by more than four sides

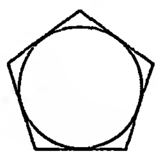
A Polygon of	*five* sides.	called a	**Pentagon,**
,,	*six* sides	,,	**Hexagon,**
,,	*seven* sides	,,	**Heptagon,**
,,	*eight* sides	,,	**Octagon,**
,,	*ten* sides	,,	**Decagon,**
,,	*twelve* sides	,,	**Dodecagon,**
,,	*fifteen* sides	,,	**Quindecagon.**

2. A Polygon is **Regular** when all its sides are equal, and all its angles are equal.

3. A rectilineal figure is said to be **inscribed in a circle**, when all its angular points are on the circumference of the circle; and a circle is said to be **circumscribed about** a rectilineal figure, when the circumference of the circle passes through all the angular points of the figure.

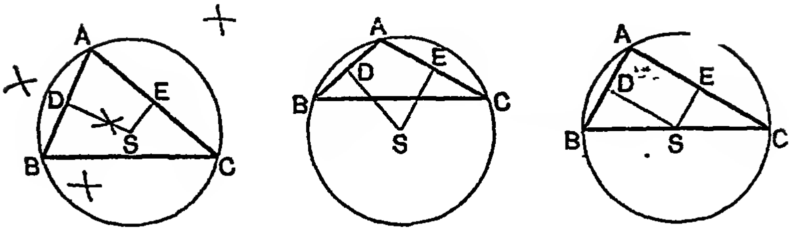


4. A circle is said to be **inscribed in** a rectilineal figure, when the circumference of the circle is touched by each side of the figure; and a rectilineal figure is said to be **circumscribed about** a circle, when each side of the figure is a tangent to the circle.



Problem 25.

To circumscribe a circle about a given triangle.



Let ABC be the triangle, about which a circle is to be drawn.

Construction. Bisect AB and AC at rt. angles by DS and ES, meeting at S. *Prob.* 2.

Then S is the centre of the required circle.

Proof. Now every point in DS is equidistant from A and B; *Prob.* 14.

and every point in ES is equidistant from A and C;

∴ S is equidistant from A, B, and C.

With centre S, and radius SA describe a circle; this will pass through B and C, and is, therefore, the required circum-circle.

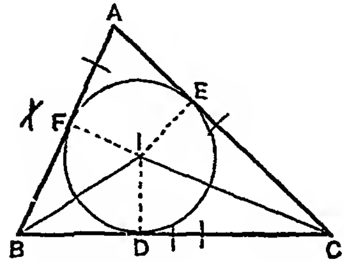
Obs. It will be found that if the given triangle is acute-angled, the centre of the circum-circle falls within it: if it is a right-angled triangle, the centre falls on the hypotenuse: if it is an obtuse-angled triangle, the centre falls without the triangle.

Note. From page 94 it is seen that if S is joined to the middle point of BC, then the joining line is perpendicular to BC.

Hence the perpendiculars drawn to the sides of a triangle from their middle points are concurrent, the point of intersection being the centre of the circle circumscribed about the triangle.

Problem 26.

To inscribe a circle in a given triangle.



Let ABC be the triangle, in which a circle is to be inscribed.

Construction. Bisect the ∠ˢ ABC, ACB by the st. lines BI, CI, which intersect at I. *Prob.* 1.

Then I is the centre of the required circle.

Proof. From I draw ID, IE, IF perp. to BC, CA, AB.

Then every point in BI is equidistant from BC, BA; *Prob.* 15.

∴ ID = IF.

And every point in CI is equidistant from CB, CA;

∴ ID = IE.

∴ ID, IE, IF are all equal.

With centre I and radius ID draw a circle;

this will pass through the points E and F.

Also the circle will touch the sides BC, CA, AB,

because the angles at D, E, F are right angles.

∴ the ⊙ DEF is inscribed in the △ ABC.

Note. From II., p. 96 it is seen that if AI is joined, then AI bisects the angle BAC: hence it follows that

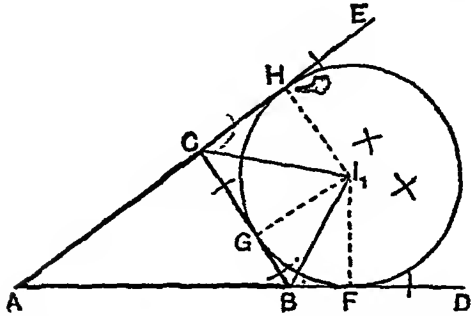
The bisectors of the angles of a triangle are concurrent, the point of intersection being the centre of the inscribed circle.

Definition.

A circle which touches one side of a triangle and the other two sides produced is called an escribed circle of the triangle.

PROBLEM 27.

To draw an escribed circle of a given triangle.



Let ABC be the given triangle of which the sides AB, AC are produced to D and E.

It is required to describe a circle touching BC, *and* AB, AC *produced.*

Construction. Bisect the $\angle^s$ CBD, BCE by the st. lines BI_1, CI_1 which intersect at I_1.

Then I_1 is the centre of the required circle.

Proof. From I_1 draw I_1F, I_1G, I_1H perp. to AD, BC, AE.

Then every point in BI_1 is equidistant from BD, BC; *Prob.* 15.

$\therefore$ $I_1F = I_1G$.

Similarly $I_1G = I_1H$.

$\therefore$ I_1F, I_1G, I_1H are all equal.

With centre I_1 and radius I_1F describe a circle;
this will pass through the points G and H.

Also the circle will touch AD, BC, and AE,
because the angles at F, G, H are rt. angles.

$\therefore$ the $\odot$FGH is an escribed circle of the $\triangle$ABC.

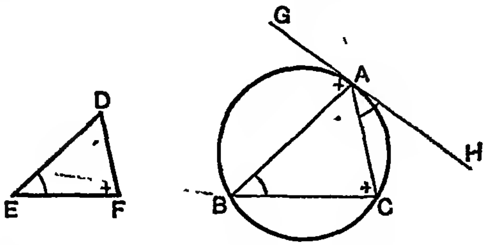
NOTE 1. It is clear that every triangle has three escribed circles. Their centres are known as the Ex-centres.

NOTE 2. It may be shewn, as in II., page 96, that if AI_1 is joined, then AI_1 bisects the angle BAC: hence it follows that

The bisectors of two exterior angles of a triangle and the bisector of the third angle are concurrent, the point of intersection being the centre of an escribed circle.

PROBLEM 28.

In a given circle to inscribe a triangle equiangular to a given triangle.



Let ABC be the given circle, and DEF the given triangle.

Analysis. A △ABC, equiangular to the △DEF, is inscribed in the circle, if from any point A on the O^{ce} two chords AB, AC can be so placed that, on joining BC, the ∠B = the ∠E, and the ∠C = the ∠F; for then the ∠A = the ∠D. *Theor.* 16.

Now the ∠B, in the segment ABC, suggests the *equal* angle between the chord AC and the tangent at its extremity (*Theor.* 49.); so that, if at A we draw the tangent GAH,

then the ∠HAC = the ∠E;

and similarly, the ∠GAB = the ∠F.

Reversing these steps, we have the following construction.

Construction. At any point A on the O^{ce} of the ⊙ABC draw the tangent GAH. *Prob.* 22.

At A make the ∠GAB equal to the ∠F,

and make the ∠HAC equal to the ∠E.

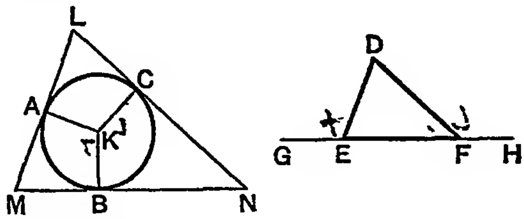
Join BC.

Then ABC is the required triangle.

NOTE. In drawing the figure on a larger scale the student should shew the construction lines for the tangent GAH and for the angles GAB, HAC. A similar remark applies to the next Problem.

PROBLEM 29.

About a given circle to circumscribe a triangle equiangular to a given triangle.



Let ABC be the given circle; and DEF the given triangle.

Analysis. Suppose LMN to be a circumscribed triangle in which the $\angle$ M = the $\angle$ E, the $\angle$ N = the $\angle$ F, and consequently, the $\angle$ L = the $\angle$ D.

Let us consider the radii KA, KB, KC, drawn to the points of contact of the sides; for the tangents LM, MN, NL could be drawn if we knew the relative positions of KA, KB, KC, that is, if we knew the $\angle$s BKA, BKC.

Now from the quad! BKAM, since the $\angle$s B and A are rt. $\angle$s,

the $\angle$ BKA $= 180° - \text{M} = 180° - \text{E}$;

similarly the $\angle$ BKC $= 180° - \text{N} = 180° - \text{F}$.

Hence we have the following construction.

Construction. Produce EF both ways to G and H.

Find K the centre of the $\odot$ ABC,
and draw any radius KB.
At K make the $\angle$ BKA equal to the $\angle$ DEG;
and make the $\angle$ BKC equal to the $\angle$ DFH.

Through A, B, C draw LM, MN, NL perp. to KA, KB, KC.
Then LMN is the required triangle.

[The student should now arrange the proof synthetically.]

EXERCISES.

On Circles and Triangles.

(Inscriptions and Circumscriptions.)

1. In a circle of radius 5 cm. inscribe an equilateral triangle; and about the same circle circumscribe a second equilateral triangle. In each case state and justify your construction.

2. Draw an equilateral triangle on a side of 8 cm., and find by calculation and measurement (to the nearest millimetre) the radii of the inscribed, circumscribed, and escribed circles.

Explain why the second and third radii are respectively double and treble of the first.

Draw triangles from the following data:

$$\text{(i) } a=2{\cdot}5'', \quad B=66^\circ, \quad C=50^\circ;$$
$$\text{(ii) } a=2{\cdot}5'', \quad B=72^\circ, \quad C=44^\circ;$$
$$\text{(iii) } a=2{\cdot}5'', \quad B=41^\circ, \quad C=23^\circ.$$

Circumscribe a circle about each triangle, and measure the radii to the nearest hundredth of an inch. Account for the three results being the same, by comparing the vertical angles.

4. In a circle of radius 4 cm. inscribe an equilateral triangle. Calculate the length of its side to the nearest millimetre; and verify by measurement.

Find the area of the inscribed equilateral triangle, and show that it is one quarter of the circumscribed equilateral triangle.

5. In the triangle ABC, if I is the centre, and r the length of the radius of the in-circle, show that

$$\triangle IBC=\tfrac{1}{2}ar; \quad \triangle ICA=\tfrac{1}{2}br; \quad \triangle IAB=\tfrac{1}{2}cr.$$

Hence prove that $\triangle ABC=\frac{1}{2}(a+b+c)r.$

Verify this formula by measurements for a triangle whose sides are 9 cm., 8 cm., and 7 cm.

6. If r_1 is the radius of the ex-circle opposite to A, prove that

$$\triangle ABC=\tfrac{1}{2}(b+c-a)r_1.$$

If $a=5$ cm., $b=4$ cm., $c=3$ cm., verify this result by measurement.

7. Find by measurement the circum-radius of the triangle ABC in which $a=6{\cdot}3$ cm., $b=3{\cdot}0$ cm., and $c=5{\cdot}1$ cm.

Draw and measure the perpendiculars from A, B, C to the opposite sides. If their lengths are represented by p_1, p_2, p_3, verify the following statement:

$$\text{circum-radius}=\frac{bc}{2p_1}=\frac{ca}{2p_2}=\frac{ab}{2p_3}.$$

EXERCISES.

On Circles and Squares.

(Inscriptions and Circumscriptions.)

1. Draw a circle of radius 1·5″, and find a construction for inscribing a square in it.

Calculate the length of the side to the nearest hundredth of an inch, and verify by measurement.

Find the area of the inscribed square.

2. Circumscribe a square about a circle of radius 1·5″, shewing all lines of construction.

Prove that the area of the square circumscribed about a circle is double that of the inscribed square.

3. Draw a square on a side of 7·5 cm., and state a construction for inscribing a circle in it.

Justify your construction by considerations of symmetry.

4. Circumscribe a circle about a square whose side is 6 cm.

Measure the diameter to the nearest millimetre, and test your drawing by calculation.

5. In a circle of radius 1·8″ inscribe a rectangle of which one side measures 3·0″. Find the approximate length of the other side.

Of all rectangles inscribed in the circle shew that the square has the greatest area.

6. A square and an equilateral triangle are inscribed in a circle. If a and b denote the lengths of their sides, shew that

$$3a^2=2b^2.$$

7. ABCD is a square inscribed in a circle, and P is any point on the arc AD: shew that the side AD subtends at P an angle three times as great as that subtended at P by any one of the other sides.

(Problems. State your construction, and give a theoretical proof.)

8. Circumscribe a rhombus about a given circle.

9. Inscribe a square in a given square ABCD, so that one of its angular points shall be at a given point X in AB.

10. In a given square inscribe the square of minimum area.

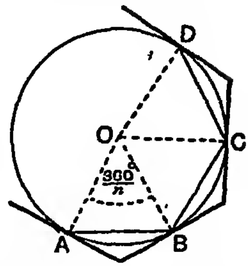
11. Describe (i) a circle, (ii) a square about a given rectangle.

12. Inscribe (i) a circle, (ii) a square in a given quadrant.

ON CIRCLES AND REGULAR POLYGONS.

PROBLEM 30.

To draw a regular polygon (i) in (ii) about *a given circle.*



Let AB, BC, CD, ... be consecutive sides of a regular polygon inscribed in a circle whose centre is O.

Then AOB, BOC, COD, ... are congruent isosceles triangles. And if the polygon has n sides, each of the

$\angle$s AOB, BOC, COD, ... $= \frac{360°}{n}$.

(i) Thus to inscribe a polygon of n sides in a given circle, draw an angle AOB at the centre equal to $\frac{360°}{n}$. This gives the length of a side AB; and chords equal to AB may now be set off round the circumference. The resulting figure will clearly be equilateral and equiangular.

(ii) To circumscribe a polygon of n sides about the circle, the points A, B, C, D, ... must be determined as before, and tangents drawn to the circle at these points. The resulting figure may readily be proved equilateral and equiangular.

NOTE. This method gives a *strict geometrical construction* only when the angle $\frac{360°}{n}$ can be drawn with ruler and compasses.

EXERCISES.

1. Give strict constructions for inscribing in a circle (radius 4 cm.) (i) a regular hexagon; (ii) a regular octagon; (iii) a regular dodecagon.

2. About a circle of radius 1·5″ circumscribe

(i) a regular hexagon; (ii) a regular octagon.

Test the constructions by measurement, and justify them by proof.

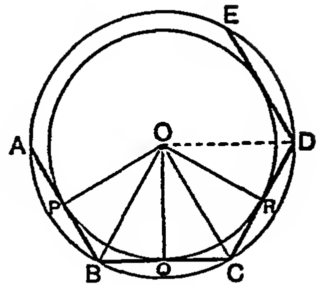
3. An equilateral triangle and a regular hexagon are inscribed in a given circle, and a and b denote the lengths of their sides: prove that

(i) area of triangle $= \frac{1}{2}$(area of hexagon); (ii) $a^2 = 3b^2$.

4. By means of your protractor inscribe a regular heptagon in a circle of radius 2″. Calculate and measure one of its angles; and measure the length of a side.

PROBLEM 31.

To draw a circle (i) in (ii) about *a regular polygon.*



Let AB, BC, CD, DE, ... be consecutive sides of a regular polygon of n sides.

Bisect the $\angle^s$ ABC, BCD by BO, CO meeting at O.

Then O is the centre both of the inscribed and circumscribed circle.

Outline of Proof. Join OD; and from the congruent $\triangle^s$ OCB, OCD, shew that OD bisects the $\angle$ CDE. Hence we conclude that

All the bisectors of the angles of the polygon meet at O.

(i) Prove that OB = OC = OD = ... ; from Theorem 6.
Hence O is the circum-centre.

(ii) Draw OP, OQ, OR, ... perp. to AB, BC, CD, ...
Prove that OP = OQ = OR = ... ; from the congruent $\triangle^s$ OBP, OBQ, ...
Hence O is the in-centre.

EXERCISES.

1. Draw a regular hexagon on a side of 2·0″. Draw the inscribed and circumscribed circles. Calculate and measure their diameters to the nearest hundredth of an inch.

2. Shew that the area of a regular hexagon inscribed in a circle is three-fourths of that of the circumscribed hexagon.

Find the area of a hexagon inscribed in a circle of radius 10 cm. to the nearest tenth of a sq. cm.

3. If ABC is an isosceles triangle inscribed in a circle, having each of the angles B and C double of the angle A; shew that BC is a side of a regular pentagon inscribed in the circle.

4. On a side of 4 cm. construct (without protractor)
(i) a regular hexagon; (ii) a regular octagon
In each case find the approximate area of the figure.

THE CIRCUMFERENCE OF A CIRCLE.

By experiment and measurement it is found that the length of the circumference of a circle is roughly $3\frac{1}{7}$ times the length of its diameter: that is to say

$$\frac{\textit{circumference}}{\textit{diameter}} = 3\tfrac{1}{7} \text{ nearly};$$

and it can be proved that this is the same for all circles.

A more correct value of this ratio is found by theory to be 3·1416; while correct to 7 places of decimals it is 3·1415926. Thus the value $3\frac{1}{7}$ (or 3·1428) is too great, and correct to 2 places only.

The ratio which the circumference of any circle bears to its diameter is denoted by the Greek letter π; so that

$$\textit{circumference} = \textit{diameter} \times \pi.$$

Or, if r denotes the radius of the circle,

$$\textit{circumference} = 2r \times \pi = 2\pi r;$$

where to π we are to give one of the values $3\frac{1}{7}$, 3·1416, or 3·1415926, according to the degree of accuracy required in the final result.

NOTE. The theoretical methods by which π is evaluated to any required degree of accuracy cannot be explained at this stage, but its value may be easily verified by experiment to two decimal places.

For example: round a cylinder wrap a strip of paper so that the ends overlap. At any point in the overlapping area prick a pin through both folds. Unwrap and straighten the strip, then measure the distance between the pin holes: this gives the length of the circumference. Measure the diameter, and divide the first result by the second.

Ex. 1. From these data find and record the value of π.

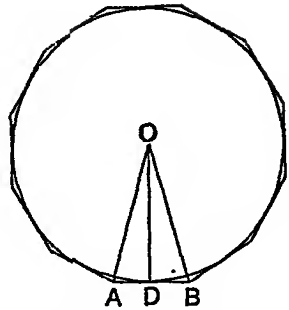
Find the mean of the three results.

CIRCUMFERENCE.	DIAMETER.	VALUE OF π.
16·0 cm.	5·1 cm.	
8·8″	2·8″	
13·5″	4·3″	

Ex. 2. A fine thread is wound evenly round a cylinder, and it is found that the length required for 20 complete turns is 75·4″. The diameter of the cylinder is 1·2″: find roughly the value of π.

Ex. 3. A bicycle wheel, 28″ in diameter, makes 400 revolutions in travelling over 977 yards. From this result estimate the value of π.

THE AREA OF A CIRCLE.



Let AB be a side of a polygon of n sides circumscribed about a circle whose centre is O and radius r. Then we have

Area of polygon

$= n . \triangle \text{AOB}$

$= n . \frac{1}{2}\text{AB} \times \text{OD}$

$= \frac{1}{2} . n\text{AB} \times r$

$= \frac{1}{2}$ (*perimeter of polygon*) $\times r$;

and this is true however many sides the polygon may have.

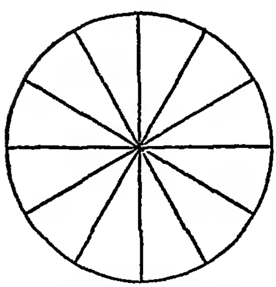
Now if the number of sides is increased without limit, the perimeter and area of the polygon may be made to differ from the circumference and area of the circle by quantities smaller than any that can be named; hence ultimately

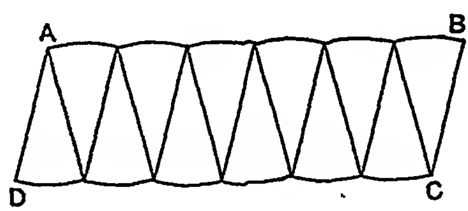
Area of circle $= \frac{1}{2}$. *circumference* $\times r$

$= \frac{1}{2} . 2\pi r \times r$

$= \pi r^2$.

ALTERNATIVE METHOD.





Suppose the circle divided into any even number of sectors having equal central angles: denote the number of sectors by n.

Let the sectors be placed side by side as represented in the diagram;

then the area of the circle = the area of the fig. ABCD;

and this is true however great n may be.

Now as the number of sectors is increased, each arc is decreased;

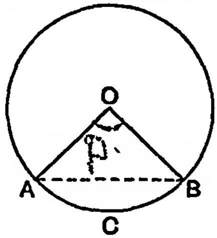
so that (i) the outlines AB, CD tend to become *straight*, and

(ii) the angles at D and B tend to become rt. angles.

Thus when n is increased without limit, the fig. ABCD ultimately becomes a *rectangle,* whose length is the *semi-circumference of the circle,* and whose breadth is its *radius.*

$$\therefore \textit{Area of circle} = \tfrac{1}{2}.\textit{circumference} \times \textit{radius}$$
$$= \tfrac{1}{2}.2\pi r \times r = \pi r^2.$$

THE AREA OF A SECTOR.



If two radii of a circle make an angle of 1°, they cut off

(i) an arc whose length $= \frac{1}{360}$ of the circumference;

and (ii) a sector whose area $= \frac{1}{360}$ of the circle;

$\therefore$ if the angle AOB contains D degrees, then

(i) *the arc* AB $= \frac{D}{360}$ *of the circumference;*

(ii) *the sector* AOB $= \frac{D}{360}$ *of the area of the circle*

$= \frac{D}{360}$ *of* ($\frac{1}{2}$ *circumference* $\times$ *radius*)

$= \frac{1}{2}$. *arc* AB $\times$ *radius.*

THE AREA OF A SEGMENT.

The area of a minor segment is found by subtracting from the corresponding sector the area of the triangle formed by the chord and the radii. Thus

Area of segment ABC = *sector* OACB − *triangle* AOB.

The area of a major segment is most simply found by subtracting the area of the corresponding minor segment from the area of the circle.

EXERCISES.

[*In each case choose the value of π so as to give a result of the assigned degree of accuracy.*]

1. Find to the nearest millimetre the circumferences of the circles whose radii are (i) 4·5 cm. (ii) 100 cm.

2. Find to the nearest hundredth of a square inch the areas of the circles whose radii are (i) 2·3″. (ii) 10·6″.

3. Find to two places of decimals the circumference and area of a circle inscribed in a square whose side is 3·6 cm.

4. In a circle of radius 7·0 cm. a square is described: find to the nearest square centimetre the difference between the areas of the circle and the square.

5. Find to the nearest hundredth of a square inch the area of the circular ring formed by two concentric circles whose radii are 5·7″ and 4·3″.

6. Shew that the area of a ring lying between the circumferences of two concentric circles is equal to the area of a circle whose radius is the length of a tangent to the inner circle from any point on the outer.

7. A rectangle whose sides are 8·0 cm. and 6·0 cm. is inscribed in a circle. Calculate to the nearest tenth of a square centimetre the total area of the four segments outside the rectangle.

8. Find to the nearest tenth of an inch the side of a square whose area is equal to that of a circle of radius 5″.

9. A circular ring is formed by the circumference of two concentric circles. The area of the ring is 22 square inches, and its width is 1·0″; taking π as $\frac{22}{7}$, find approximately the radii of the two circles.

10. Find to the nearest hundredth of a square inch the difference between the areas of the circumscribed and inscribed circles of an equilateral triangle each of whose sides is 4″.

11. Draw on squared paper two circles whose centres are at the points (1·5″, 0) and (0, ·8″), and whose radii are respectively ·7″ and 1·0″. Prove that the circles touch one another, and find approximately their circumferences and areas.

12. Draw a circle of radius 1·0″ having the point (1·6″, 1·2″) as centre. Also draw two circles with the origin as centre and of radii 1·0″ and 3·0″ respectively. Shew that each of the last two circles touches the first.

EXERCISES.

ON THE INSCRIBED, CIRCUMSCRIBED, AND ESCRIBED CIRCLES OF A TRIANGLE.

(*Theoretical.*)

1. Describe a circle to touch two parallel straight lines and a third straight line which meets them. Show that two such circles can be drawn, and that they are equal.

2. *Triangles which have equal bases and equal vertical angles have equal circumscribed angles.*

3. ABC is a triangle, and I, S are the centres of the inscribed and circumscribed circles; if A, I, S are collinear, shew that AB = AC.

4. The sum of the diameters of the inscribed and circumscribed circles of a right-angled triangle is equal to the sum of the sides containing the right angle.

5. If the circle inscribed in the triangle ABC touches the sides at D, E, F; show that the angles of the triangle DEF are respectively

$$90-\frac{A}{2},\ 90-\frac{B}{2},\ 90-\frac{C}{2}.$$

6. If I is the centre of the circle inscribed in the triangle ABC, and I_1 the centre of the escribed circle which touches BC; shew that I, B, I_1, C are concyclic.

7. In any triangle the difference of two sides is equal to the difference of the segments into which the third side is divided at the point of contact of the inscribed circle.

8. In the triangle ABC, I and S are the centres of the inscribed and circumscribed circles: shew that IS subtends at A an angle equal to half the difference of the angles at the base of the triangle.

Hence shew that if AD is drawn perpendicular to BC, then AI is the bisector of the angle DAS.

9. The diagonals of a quadrilateral ABCD intersect at O: show that the centres of the circles circumscribed about the four triangles AOB, BOC, COD, DOA are at the angular points of a parallelogram.

10. In any triangle ABC, if I is the centre of the inscribed circle, and if AI is produced to meet the circumscribed circle at O; shew that O is the centre of the circle circumscribed about the triangle BIC.

11. Given the base, altitude, and the radius of the circumscribed circle; construct the triangle.

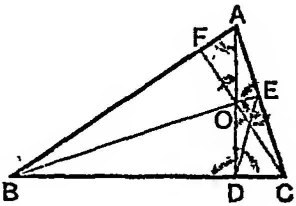
12. Three circles whose centres are A, B, C touch one another externally two by two at D, E, F: shew that the inscribed circle of the triangle ABC is the circumscribed circle of the triangle DEF.

THEOREMS AND EXAMPLES ON CIRCLES AND TRIANGLES.

THE ORTHOCENTRE OF A TRIANGLE.

1. *The perpendiculars drawn from the vertices of a triangle to the opposite sides are concurrent.*

In the △ABC, let AD, BE be the perp^s drawn from A and B to the opposite sides; and let them intersect at O.



Join CO; and produce it to meet AB at F.

It is required to shew that CF *is perp. to* AB.

Join DE.

Then, because the ∠^s OEC, ODC are rt. angles,

∴ the points O, E, C, D are concyclic:
∴ the ∠DEC = the ∠DOC, in the same segment;
= the vert. opp. ∠FOA.

Again, because the ∠^s AEB, ADB are rt. angles,
∴ the points A, E, D, B are concyclic:
∴ the ∠DEB = the ∠DAB, in the same segment.

∴ the sum of the ∠^s FOA, FAO = the sum of the ∠^s DEC, DEB
= a rt. angle:
∴ the remaining ∠AFO = a rt. angle: *Theor.* 16.
that is, CF is perp. to AB.

Hence the three perp^s AD, BE, CF meet at the point O.

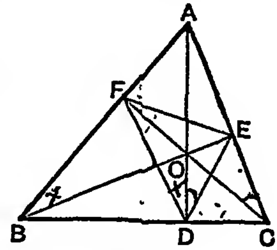
Q.E.D.

DEFINITIONS.

(i) The intersection of the perpendiculars drawn from the vertices of a triangle to the opposite sides is called its orthocentre.

(ii) The triangle formed by joining the feet of the perpendiculars is called the pedal or orthocentric triangle.

II. *In an acute-angled triangle the perpendiculars drawn from the vertices to the opposite sides bisect the angles of the pedal triangle through which they pass.*



In the acute-angled △ ABC, let AD, BE, CF be the perp^s drawn from the vertices to the opposite sides, meeting at the ortho-centre O; and let DEF be the pedal triangle.

It is required to prove that

AD, BE, CF *bisect respectively* *the* ∠^{s} FDE, DEF, EFD.

It may be shewn, as in the last theorem, that the points O, D, C, E are concyclic;

∴ the ∠ODE = the ∠OCE, in the same segment.

Similarly the points O, D, B, F are concyclic;

∴ the ∠ODF = the ∠OBF, in the same segment.

But the ∠OCE = the ∠OBF, each being the comp^t of the ∠BAC.

∴ the ∠ODE = the ∠ODF.

Similarly it may be shown that the ∠^{s} *DEF, EFD are bisected by* BE and CF. Q.E.D.

Corollary. (i) *Every two sides of the pedal triangle are equally inclined to that side of the original triangle in which they meet.*

For the ∠EDC = the comp^t of the ∠ODE
= the comp^t of the ∠OCE
= the ∠BAC.

Similarly it may be shewn that the ∠FDB = the ∠BAC,

∴ the ∠EDC = the ∠FDB = the ∠A.

In like manner it may be proved that

the ∠DEC = the ∠FEA = the ∠B,
and the ∠DFB = the ∠EFA = the ∠C.

Corollary. (ii) *The triangles* DEC, AEF, DBF *are equiangular to one another and to the triangle* ABC.

Note. If the angle BAC is *obtuse*, then the perpendiculars BE, CF bisect *externally* the corresponding angles of the pedal triangle.

EXERCISES.

1. *If* O *is the orthocentre of the triangle* ABC *and if the perpendicular* AD *is produced to meet the circum-circle in* G, *prove that* OD = DG.

2. *In an acute-angled triangle the three sides are the external bisectors of the angles of the pedal triangle: and in an obtuse-angled triangle the sides containing the obtuse angle are the internal bisectors of the corresponding angles of the pedal triangle.*

3. *If* O *is the orthocentre of the triangle* ABC, *shew that the angles* BOC, BAC *are supplementary.*

4. *If* O *is the orthocentre of the triangle* ABC, *then any one of the four points* O, A, B, C *is the orthocentre of the triangle whose vertices are the other three.*

5. *The three circles which pass through two vertices of a triangle and its orthocentre are each equal to the circum-circle of the triangle.*

6. D, E are taken on the circumference of a semi-circle described on a given straight line AB: the chords AD, BE and AE, BD intersect (produced if necessary) at F and G: shew that FG is perpendicular to AB.

7. ABC is a triangle, O is its orthocentre, and AK a diameter of the circum-circle: shew that BOCK is a parallelogram.

8. The orthocentre of a triangle is joined to the middle point of the base, and the joining line is produced to meet the circum-circle: prove that it will meet it at the same point as the diameter which passes through the vertex.

9. The perpendicular from the vertex of a triangle on the base, and the straight line joining the orthocentre to the middle point of the base, are produced to meet the circum-circle at P and Q: shew that PQ is parallel to the base.

10. *The distance of each vertex of a triangle from the orthocentre is double of the perpendicular drawn from the centre of the circum-circle to the opposite side.*

11. Three circles are described each passing through the orthocentre of a triangle and two of its vertices: shew that the triangle formed by joining their centres is equal in all respects to the original triangle.

12. Construct a triangle, having given a vertex, the orthocentre, and the centre of the circum-circle.

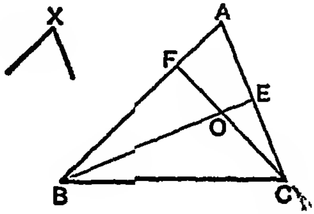
LOCI.

III. *Given the base and vertical angle of a triangle, find the locus of its orthocentre.*

Let BC be the given base, and X the given angle; and let BAC be any triangle on the base BC, having its vertical $\angle$A equal to the $\angle$X.

Draw the perp^s BE, CF, intersecting at the orthocentre O.

It is required to find the locus of O.



Proof. Since the $\angle^s$ OFA, OEA are rt. angles,

$\therefore$ the points O, F, A, E are concyclic;

$\therefore$ the $\angle$FOE is the supplement of the $\angle$A:

$\therefore$ the vert. opp. $\angle$BOC is the supplement of the $\angle$A.

But the $\angle$A is constant, being always equal to the $\angle$X;

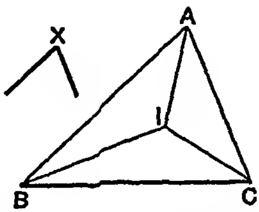
$\therefore$ its supplement is constant;

that is, the $\triangle$BOC has a fixed base, and constant vertical angle; hence the locus of its vertex O is the arc of a segment of which BC is the chord.

IV. *Given the base and vertical angle of a triangle, find the locus of the in-centre.*

Let BAC be any triangle on the given base BC, having its vertical angle equal to the given $\angle$X; and let AI, BI, CI be the bisectors of its angles. Then I is the in-centre.

It is required to find the locus of I.



Proof. Denote the angles of the $\triangle$ABC by A, B, C; and let the $\angle$BIC be denoted by I.

Then from the $\triangle$BIC,

(i) $\quad$ I$+\frac{1}{2}$B$+\frac{1}{2}$C$=$two rt. angles; $\quad$ *Theor.* 16

and from the $\triangle$ABC,

A$+$B$+$C$=$two rt. angles;

(ii) so that $\frac{1}{2}$A$+\frac{1}{2}$B$+\frac{1}{2}$C$=$one rt. angle.

$\therefore$, taking the differences of the equals in (i) and (ii),

I$-\frac{1}{2}$A$=$one rt. angle:

or, $\quad$ I$=$one rt. angle$+\frac{1}{2}$A.

But A is constant, being always equal to the $\angle$X;

$\therefore$ I is constant:

$\therefore$ the locus of I is the arc of a segment on the fixed chord BC.

EXERCISES ON LOCI.

1. Given the base BC and the vertical angle A of a triangle; find the locus of the ex-centre opposite A.

2. Through the extremities of a given straight line AB any two parallel straight lines AP, BQ are drawn; find the locus of the intersection of the bisectors of the angles PAB, QBA.

3. Find the locus of the middle points of chords of a circle drawn through a fixed point.

Distinguish between the cases when the given point is within, on, or without the circumference.

4. Find the locus of the points of contact of tangents drawn from a fixed point to a system of concentric circles.

5. Find the locus of the intersection of straight lines which pass through two fixed points on a circle and intercept on its circumference an arc of constant length.

6. A and B are two fixed points on the circumference of a circle, and PQ is any diameter: find the locus of the intersection of PA and QB.

7. BAC is any triangle described on the fixed base BC and having a constant vertical angle; and BA is produced to P, so that BP is equal to the sum of the sides containing the vertical angle: find the locus of P.

8. AB is a fixed chord of a circle, and AC is a moveable chord passing through A: if the parallelogram CB is completed, find the locus of the intersection of its diagonals.

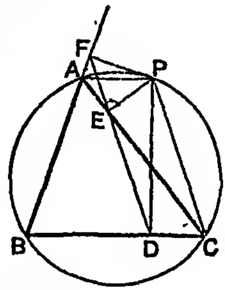
9. A straight rod PQ slides between two rulers placed at right angles to one another, and from its extremities PX, QX are drawn perpendicular to the rulers: find the locus of X.

10. Two circles intersect at A and B, and through P, any point on the circumference of one of them, two straight lines PA, PB are drawn, and produced if necessary, to cut the other circle at X and Y: find the locus of the intersection of AY and BX.

11. Two circles intersect at A and B; HAK is a fixed straight line drawn through A and terminated by the circumferences, and PAQ is any other straight line similarly drawn: find the locus of the intersection of HP and QK.

SIMSON'S LINE.

V. *The feet of the perpendiculars drawn to the three sides of a triangle from any point on its circum-circle are collinear.*



Let P be any point on the circum-circle of the △ ABC; and let PD, PE, PF be the perps. drawn from P to the sides.

It is required to prove that the points D, E, F *are collinear.*

Join FE and ED:

then FE and ED will be shewn to be in the same straight line.

Join PA, PC.

Proof. Because the ∠ˢ PEA, PFA are rt. angles,

∴ the points P, E, A, F are concyclic:

∴ the ∠PEF = the ∠PAF, in the same segment
= the suppᵗ of the ∠PAB
= the ∠PCD,

since the points A, P, C, B are concyclic.

Again because the ∠ˢ PEC, PDC are rt. angles,

∴ the points P, E, D, C are concyclic.

∴ the ∠PED = the suppᵗ of the ∠PCD
= the suppᵗ of the ∠PEF.

∴ FE and ED are in one st. line.

Obs. The line FED is known as the **Pedal** or **Simson's Line** of the triangle ABC for the point P.

EXERCISES.

1. From any point P on the circum-circle of the triangle ABC, perpendiculars PD, PF are drawn to BC and AB: if FD, or FD produced, cuts AC at E, shew that PE is perpendicular to AC.

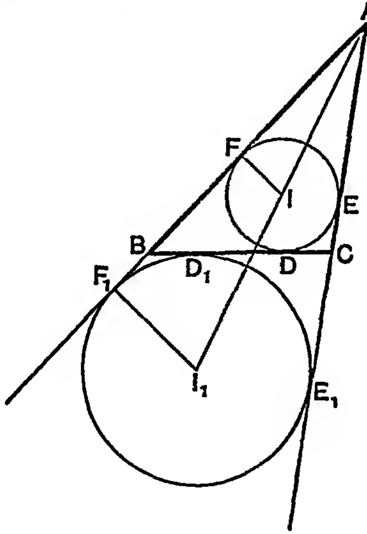
2. Find the locus of a point which moves so that if perpendiculars are drawn from it to the sides of a given triangle, their feet are collinear.

3. ABC and AB′C′ are two triangles with a common angle, and their circum-circles meet again at P; shew that the feet of perpendiculars drawn from P to the lines AB, AC, BC, B′C′ are collinear.

4. A triangle is inscribed in a circle, and any point P on the circumference is joined to the orthocentre of the triangle: shew that this joining line is bisected by the pedal of the point P.

THE TRIANGLE AND ITS CIRCLES.

VI. D, E, F *are the points of contact of the inscribed circle of the triangle* ABC, *and* D_1, E_1, F_1 *the points of contact of the escribed circle, which touches* BC *and the other sides produced:* a, b, c *denote the length of the sides* BC, CA, AB; s *the semi-perimeter of the triangle, and* r, r_1 *the radii of the inscribed and escribed circles.*



Prove the following equalities.

(i) AE = AF $= s - a$,
BD = BF $= s - b$,
CD = CE $= s - c$.

(ii) $AE_1 = AF_1 = s$.

(iii) $CD_1 = CE_1 = s - b$,
$BD_1 = BF_1 = s - c$.

(iv) CD $= BD_1$, and BD $= CD_1$.

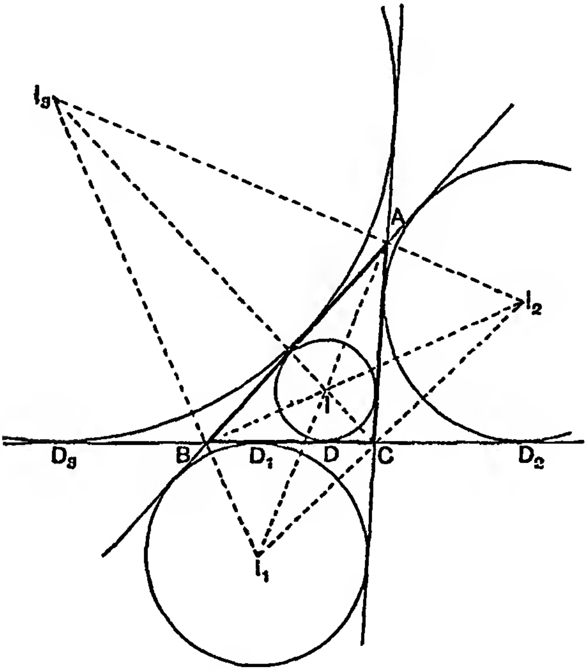
(v) $EE_1 = FF_1 = a$.

(vi) The area of the $\triangle$ ABC $= rs$
$= r_1(s - a)$.

(vii) Draw the above figure in the case when C is a right angle, and prove that

$$r = s - c; \quad r_1 = s - b.$$

VII. *In the triangle* ABC, I *is the centre of the inscribed circle, and* I_1, I_2, I_3 *the centres of the escribed circles touching respectively the sides* BC, CA, AB *and the other sides produced.*



Prove the following properties:

(i) *The points* A, I, I_1 *are collinear: so are* B, I, I_2; *and* C, I, I_3.

(ii) *The points* I_2, A, I_3 *are collinear; so are* I_3, B, I_1; *and* I_1, C, I_2.

(iii) *The triangles* BI_1C, CI_2A, AI_3B *are equiangular to one another.*

(iv) *The triangle* $I_1I_2I_3$ *is equiangular to the triangle formed by joining the points of contact of the inscribed circle.*

(v) *Of the four points* I, I_1, I_2, I_3, *each is the orthocentre of the triangle whose vertices are the other three.*

(vi) *The four circles, each of which passes through three of the points* I, I_1, I_2, I_3, *are all equal.*

EXERCISES.

1. With the figure given on page 214 shew that if the circles whose centres are I, I_1, I_2, I_3 touch BC at D, D_1, D_2, D_3, then

(i) $DD_2 = D_1D_3 = b$. (ii) $DD_3 = D_1D_2 = c$.
(iii) $D_2D_3 = b + c$. (iv) $DD_1 = b \sim c$.

2. *Shew that the orthocentre and vertices of a triangle are the centres of the inscribed and escribed circles of the pedal triangle.*

3. *Given the base and vertical angle of a triangle, find the locus of the centre of the escribed circle which touches the base.*

4. *Given the base and vertical angle of a triangle, shew that the centre of the circum-circle is fixed.*

5. Given the base BC, and the vertical angle A of the triangle, find the locus of the centre of the escribed circle which touches AC.

6. Given the base, the vertical angle, and the point of contact with the base of the in-circle; construct the triangle.

7. Given the base, the vertical angle, and the point of contact with the base, or base produced, of an escribed circle; construct the triangle.

8. I *is the centre of the circle inscribed in a triangle, and* I_1, I_2, I_3 *the centres of the escribed circles; shew that* II_1, II_2, II_3 *are bisected by the circumference of the circum-circle.*

9. ABC is a triangle, and I_2, I_3 the centres of the escribed circles which touch AC, and AB respectively: shew that the points B, C, I_2, I_3 lie upon a circle whose centre is on the circumference of the circum-circle of the triangle ABC.

10. With three given points as centres describe three circles touching one another two by two. How many solutions will there be?

11. Given the centres of the three escribed circles; construct the triangle.

12. Given the centre of the inscribed circle, and the centres of two escribed circles; construct the triangle.

13. Given the vertical angle, perimeter, and radius of the inscribed circle; construct the triangle.

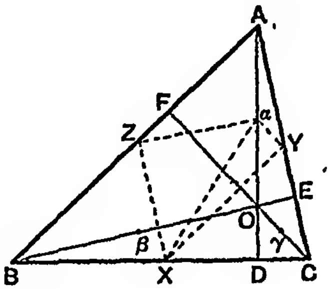
14. Given the vertical angle, the radius of the inscribed circle, and the length of the perpendicular from the vertex to the base; construct the triangle.

15. In a triangle ABC, I is the centre of the inscribed circle; shew that the centres of the circles circumscribed about the triangles BIC, CIA, AIB lie on the circumference of the circle circumscribed about the given triangle.

THE NINE-POINTS CIRCLE.

VIII. *In any triangle the middle points of the sides, the feet of the perpendiculars from the vertices to the opposite sides, and the middle points of the lines joining the orthocentre to the vertices are concyclic.*

In the $\triangle$ ABC, let X, Y, Z be the middle points of the sides BC, CA, AB; let D, E, F be the feet of the perp[s] to these sides from A, B, C; let O be the orthocentre, and α, β, γ the middle points of OA, OB, OC.



It is required to prove that the nine points X, Y, Z, D, E, F, α, β, γ *are concyclic.*

Join XY, XZ, Xα, Yα, Zα.

Now from the $\triangle$ ABO,
since AZ=ZB, and Aα=αO,
$\therefore$ Zα is par[l] to BO. Ex. 2, p. 64.

And from the $\triangle$ ABC, since BZ=ZA, and BX=XC,
$\therefore$ ZX is par[l] to AC.

But BO produced makes a rt. angle with AC;
$\therefore$ the $\angle$XZα is a rt. angle.

Similarly, the $\angle$XYα is a rt. angle.
$\therefore$ the points X, Z, α, Y are concyclic:
that is, α lies on the O[ce] of the circle which passes through X, Y, Z; and Xα is a diameter of this circle.

Similarly it may be shewn that β and γ lie on the O[ce] of this circle.

Again, since αDX is a rt. angle,
$\therefore$ the circle on Xα as diameter passes through D.

Similarly it may be shewn that E and F lie on the O[ce] of this circle;
$\therefore$ the points X, Y, Z, D, E, F, α, β, γ are concyclic. Q.E.D.

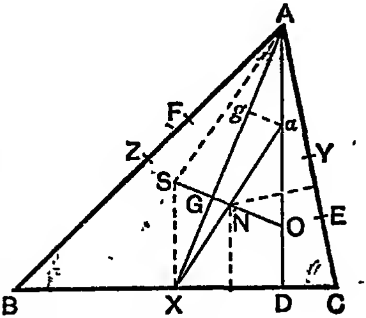
Obs. From this property the circle which passes through the middle points of the sides of a triangle is called the Nine-Points Circle; many of its properties may be derived from the fact of its being the circum-circle of the pedal triangle.

To prove that

(i) *the centre of the nine-points circle is the middle point of the straight line which joins the orthocentre to the circum-centre.*

(ii) *the radius of the nine-points circle is half the radius of the circum-circle.* in the same line

(iii) *the centroid is collinear with the circum-centre, the nine-points centre, and the orthocentre.*



In the $\triangle ABC$, let X, Y, Z be the middle points of the sides; D, E, F the feet of the perp^s; O the orthocentre; S and N the centres of the circumscribed and nine-points circles respectively.

(i) *To prove that* N *is the middle point of* SO.

It may be shewn that the perp. to XD from its middle point bisects SO; *Theor.* 22.

Similarly the perp. to EY at its middle point bisects SO:

that is, these perp^s intersect at the middle point of SO:

And since XD and EY are chords of the nine-points circle,

∴ the intersection of the lines which bisect XD and EY at rt. angles is its centre: *Theor.* 31, *Cor.* 1.

∴ the centre N is the middle point of SO. Q.E.D.

(ii) *To prove that the radius of the nine-points circle is half the radius of the circum-circle.*

By the last Proposition, $X\alpha$ is a diameter of the nine-points circle.

∴ the middle point of $X\alpha$ is its centre:

but the middle point of SO is also the centre of the nine-points circle. (*Proved.*)

Hence $X\alpha$ and SO bisect one another at N.

Then from the $\triangle^s$ SNX, $ON\alpha$,

$$\text{because}\begin{cases} SN=ON, \\ \text{and } NX=N\alpha, \\ \text{and the } \angle SNX=\text{the } \angle ON\alpha; \end{cases}$$

$$\therefore\ SX=O\alpha$$
$$=A\alpha.$$

And SX is also par^l to $A\alpha$,

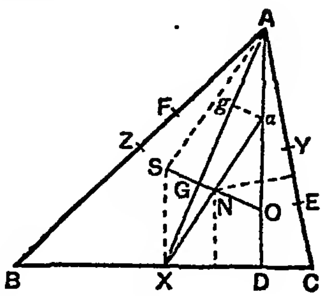
$$\therefore\ SA=X\alpha.$$

But SA is a *radius* of the circum-circle;

and $X\alpha$ is a *diameter* of the nine-points circle;

∴ the radius of the nine-points circle is half the radius of the circum-circle. [See also p. 267, Examples 2 and 3.] Q.E.D.

(iii) *To prove that the centroid is collinear with points* S, N, O.



Join AX and draw ag par^l to SO.
Let AX meet SO at G.

Then from the $\triangle$AGO, since $Aa=aO$,
and ag is par^l to OG,
$\therefore$ $Ag=gG$. Ex. 1, p. 64.

And from the $\triangle Xag$, since $aN=NX$,
and NG is par^l to ag,
$\therefore$ $gG=GX$.
$\therefore$ $AG=\frac{2}{3}$ of AX;
$\therefore$ G is the centroid of the triangle ABC.
Theor. III., *Cor.*, p. 97.
That is, the centroid is collinear with the points S, N, O. Q.E.D.

EXERCISES.

1. *Given the base and vertical angle of a triangle, find the locus of the centre of the nine-points circle.*

2. The nine-points circle of any triangle ABC, whose orthocentre is O, is also the nine-points circle of each of the triangles AOB, BOC, COA.

3. If I, I_1, I_2, I_3 are the centres of the inscribed and escribed circles of a triangle ABC, then the circle circumscribed about ABC is the nine-points circle of each of the four triangles formed by joining three of the points I, I_1, I_2, I_3.

4. All triangles which have the same orthocentre and the same circumscribed circle, have also the same nine-points circle.

5. Given the base and vertical angle of a triangle, shew that one angle and one side of the pedal triangle are constant.

6. Given the base and vertical angle of a triangle, find the locus of the centre of the circle which passes through the three escribed centres.

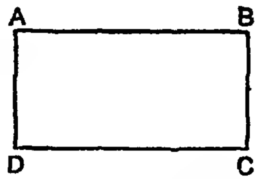
NOTE. For some other important properties of the Nine-points Circle see Ex. 54, page 310.

PART IV.

ON SQUARES AND RECTANGLES IN CONNECTION WITH THE SEGMENTS OF A STRAIGHT LINE.

THE GEOMETRICAL EQUIVALENTS OF CERTAIN ALGEBRAICAL FORMULÆ.

DEFINITIONS.



1. A rectangle ABCD is said to be contained by two adjacent sides AB, AD; for these sides fix its size and shape.

A rectangle whose adjacent sides are AB, AD is denoted by *the rect.* AB, AD; this is equivalent to the product AB . AD.

Similarly a square drawn on the side AB is denoted by *the sq. on* AB, or AB^2.

2. If a point X is taken in a straight line AB, or in AB produced, then X is said to divide AB into the two segments AX, XB; the segments being in either case *the distances of the dividing point* X *from the extremities of the given line* AB.

Fig. 1.

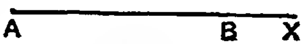


Fig. 2.

In Fig. 1, AB is said to be divided internally at X.

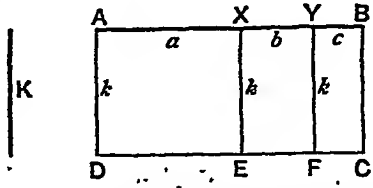
In Fig. 2, AB .... divided externally at X.

Obs. In internal division the given line AB is the *sum* of the segments AX, XB.

In external division the given line AB is the *difference* of the segments AX, XB.

THEOREM 50. [Euclid II. 1.]

If of two straight lines, one is divided into any number of parts, the rectangle contained by the two lines is equal to the sum of the rectangles contained by the undivided line and the several parts of the divided line.



Let AB and K be the two given st. lines, and let AB be divided into any number of parts AX, XY, YB, which contain respectively a, b, and c units of length; so that AB contains $a+b+c$ units.

Let the line K contain k units of length.

It is required to prove that

the rect. AB, K = rect. AX, K + rect. XY, K + rect. YB, K;

namely that

$$(a+b+c)k = \quad ak \quad + \quad bk \quad + \quad ck.$$

Construction. Draw AD perp. to AB and equal to K.

Through D draw DC par[l] to AB.

Through X, Y, B draw XE, YF, BC par[l] to AD.

Proof. The fig. AC = the fig. AE + the fig. XF + the fig. YC; and of these, by construction,

fig. AC = rect. AB, K; and contains $(a+b+c)k$ units of area;

$$\begin{cases} \text{fig. AE = rect. AX, K; and contains } ak \text{ units of area;} \\ \text{fig. XF = rect. XY, K; } \ldots\ldots\ldots\ldots\ldots \; bk \; \ldots\ldots\ldots\ldots\ldots; \\ \text{fig. YC = rect. YB, K; } \ldots\ldots\ldots\ldots\ldots \; ck \; \ldots\ldots\ldots\ldots\ldots. \end{cases}$$

Hence

the rect. AB, K = rect. AX, K + rect. XY, K + rect. YB, K;

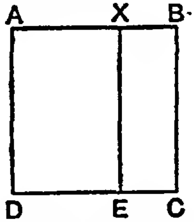
or, $\quad (a+b+c)k = \quad ak \quad + \quad bk \quad + \quad ck.$

Q.E.D.

* Corollaries. [Euclid II. 2 and 3.]

Two special cases of this Theorem deserve attention.

(i) When AB is divided only at one point X, and when the undivided line AD is equal to AB.



Then the sq. on AB = the rect. AB, AX + the rect. AB, XB.

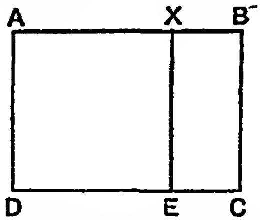
That is,

The square on the given line is equal to the sum of the rectangles contained by the whole line and each of the segments.

Or thus:

$$\begin{aligned} AB^2 &= AB \, . \, AB \\ &= AB(AX + XB) \\ &= AB \, . \, AX + AB \, . \, XB. \end{aligned}$$

(ii) When AB is divided at one point X, and when the undivided line AD is equal to one segment AX.



Then the rect. AB, AX = the sq. on AX + the rect. AX, XB.

That is,

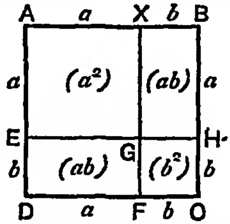
The rectangle contained by the whole line and one segment is equal to the square on that segment with the rectangle contained by the two segments.

Or thus:

$$\begin{aligned} AB \, . \, AX &= (AX + XB)AX \\ &= AX^2 + AX \, . \, XB. \end{aligned}$$

THEOREM 51. [Euclid II. 4.]

If a straight line is divided **internally** *at any point, the square on the given line is equal to the sum of the squares on the two segments* **together with** *twice the rectangle contained by the segments.*



Let AB be the given st. line divided *internally* at X; and let the segments AX, XB contain a and b units of length respectively.

Then AB is the *sum* of the segments AX, XB, and therefore contains $a+b$ units.

It is required to prove that

$$AB^2 = AX^2 + XB^2 + 2AX \cdot XB;$$

namely that

$$(a+b)^2 = a^2 + b^2 + 2ab.$$

Construction. On AB describe a square ABCD. From AD cut off AE equal to AX, or a. Then ED = XB = b. Through E and X draw EH, XF par[1] respectively to AB, AD and meeting at G.

Proof. Then the fig. AC = the figs. AG, GC + the figs. EF, XH.
And of these, by construction,
fig. AC is the sq. on AB, and contains $(a+b)^2$ units of area;

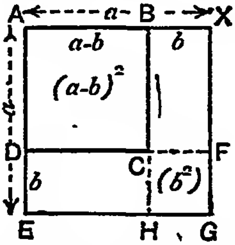
fig. AG = sq. on AX, and contains a^2 units of area;
fig. GC = sq. on XB, ... b^2 .... ;
fig. EF = rect. EG, ED
= rect. AX, XB .... ab .... ;
fig. XH = rect. GX, XB
= rect. AX, XB .... ab ....

Hence $AB^2 = AX^2 + XB^2 + 2AX \cdot XB;$
that is, $(a+b)^2 = a^2 + b^2 + 2ab.$

Q.E.D.

THEOREM 52. [Euclid II. 7.]

If a straight line is divided externally *at any point, the square on the given line is equal to the sum of the squares on the two segments* diminished by *twice the rectangle contained by the segments.*



Let AB be the given st. line divided *externally* at X; and let the segments AX, XB contain a and b units of length respectively.

Then AB is the *difference* of the segments AX, XB, and therefore contains $a-b$ units.

It is required to prove that

$$AB^2 = AX^2 + XB^2 - 2AX \cdot XB;$$

namely that $(a-b)^2 = a^2 + b^2 - 2ab.$

Construction. On AX describe a square AXGE. From AE cut off AD equal to AB, or $a-b$. Then $ED = XB = b$. Through D and B draw DF, BH par[l] respectively to AX, AE, meeting at C.

Proof. Then the fig. AC = the figs. AG, CG – the figs. EF, XH.

And of these, by construction,

fig. AC is the sq. on AB, and contains $(a-b)^2$ units of area;

fig. AG = sq. on AX, and contains a^2 units of area;
fig. CG = sq. on XB, ... b^2 ....;
fig. EF = rect. EG, ED
= rect. AX, XB .... ab ....;
fig. XH = rect. GX, XB
= rect. AX, XB .... ab . ...

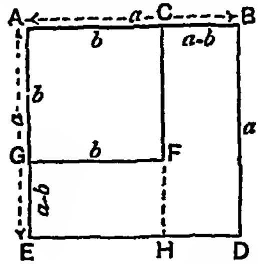
Hence $AB^2 = AX^2 + XB^2 - 2AX \cdot XB;$

that is, $(a-b)^2 = a^2 + b^2 - 2ab.$

Q.E.D.

THEOREM 53. [Euclid II. 5 and 6.]

The difference of the squares on two straight lines is equal to the rectangle contained by their sum and difference.



Let the given lines AB, AC be placed in the same st. line, and let them contain a and b units of length respectively.

It is required to prove that

$$AB^2 - AC^2 = (AB + AC)(AB - AC);$$

namely that $$a^2 - b^2 = (a+b)(a-b).$$

Construction. On AB and AC draw the squares ABDE, ACFG; and produce CF to meet ED at H.

Then $GE = CB = a - b$ units.

Proof. Now $AB^2 - AC^2$ = the sq. AD − the sq. AF

= the rect. CD + the rect. GH

$= DB \cdot BC + GF \cdot GE$

$= AB \cdot CB + AC \cdot CB$

$= (AB + AC)\,CB$

$= (AB + AC)(AB - AC).$

That is,

$$a^2 - b^2 = (a+b)(a-b).$$

Q.E.D.

COROLLARY. *If a straight line is bisected, and also divided (internally or externally) into two unequal segments, the rectangle contained by these segments is equal to the difference of the squares on half the line and on the line between the points of section.*

Fig. 1.

Fig. 2.

That is, if AB is bisected at X and also divided at Y, internally in Fig. 1, and externally in Fig. 2, then

in Fig. 1, $AY . YB = AX^2 - XY^2$;
in Fig. 2, $AY . YB = XY^2 - AX^2$.

For in the first case, $AY . YB = (AX + XY)(XB - XY)$
$= (AX + XY)(AX - XY)$
$= AX^2 - XY^2$.

The second case may be similarly proved.

EXERCISES.

1. Draw diagrams on squared paper to shew that the square on a straight line is

(i) *four*-times the square on *half* the line;
(ii) *nine*-times the square on *one-third* of the line.

2. Draw diagrams on squared paper to illustrate the following algebraical formulæ:

(i) $(x+7)^2 = x^2 + 14x + 49$.
(ii) $(a+b+c)^2 = a^2 + b^2 + c^2 + 2bc + 2ca + 2ab$.
(iii) $(a+b)(c+d) = ac + ad + bc + bd$.
(iv) $(x+7)(x+9) = x^2 + 16x + 63$.

3. In Theor. 50, Cor. (i), if AB = 4 cm., and the fig. AE = 9·6 sq. cm., find the area of the fig. XC.

4. In Theor. 50, Cor. (ii), if AX = 2·1″, and the fig. XC = 3·36 sq. in., find AB.

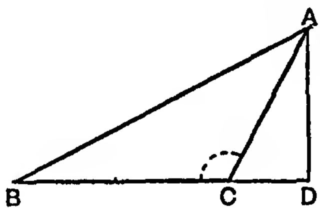
5. In Theor. 51, if the fig. AG = 36 sq. cm., and the rect. AX, XB = 24 sq. cm., find AB.

6. In Theorem 52, if the fig. AG = 9·61 sq. in., and the fig. DG = 6·51 sq. in., find AB.

[For further Examples on Theorems 50-53 see p. 230.]

THEOREM 54. [Euclid II. 12.]

In an obtuse-angled triangle, the square on the side subtending the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle **together with** *twice the rectangle contained by one of those sides and the* **projection of the other side** *upon it.*



Let ABC be a triangle obtuse-angled at C; and let AD be drawn perp. to BC produced, so that CD is the projection of the side CA on BC. [See Def. p. 63.]

It is required to prove that

$$AB^2 = BC^2 + CA^2 + 2\,BC \,.\, CD.$$

Proof. Because BD is the *sum* of the lines BC, CD,

$$\therefore\ BD^2 = BC^2 + CD^2 + 2\,BC \,.\, CD. \qquad \textit{Theor. } 51$$

To each of these equals add DA^2.

Then $BD^2 + DA^2 = BC^2 + (CD^2 + DA^2) + 2\,BC.\,CD.$

$$\left.\begin{aligned}\text{But } BD^2 + DA^2 &= AB^2\\ \text{and } CD^2 + DA^2 &= CA^2\end{aligned}\right\}, \text{ for the } \angle D \text{ is a rt. } \angle.$$

Hence $AB^2 = BC^2 + CA^2 + 2\,BC \,.\, CD.$

Q.E.D.

THEOREM 55. [Euclid II. 13.]

In every triangle the square on the side subtending an acute angle is equal to the sum of the squares on the sides containing that angle diminished by twice the rectangle contained by one of those sides and the projection of the other side *upon it.*

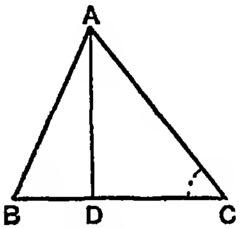


Fig. 1.

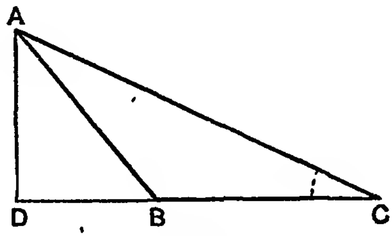


Fig. 2.

Let ABC be a triangle in which the $\angle$C is acute; and let AD be drawn perp. to BC, or BC produced; so that CD is the projection of the side CA on BC.

It is required to prove that

$$AB^2 = BC^2 + CA^2 - 2BC \cdot CD.$$

Proof. Since in both figures BD is the *difference* of the lines BC, CD,

$$\therefore \; BD^2 = BC^2 + CD^2 - 2BC \cdot CD. \qquad \textit{Theor. 52.}$$

To each of these equals add DA^2.

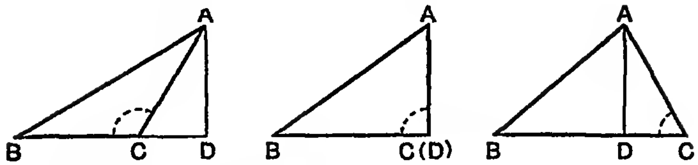
Then $BD^2 + DA^2 = BC^2 + (CD^2 + DA^2) - 2BC \cdot CD.$ ......(i)

But $\left.\begin{array}{l} BD^2 + DA^2 = AB^2 \\ \text{and } CD^2 + DA^2 = CA^2 \end{array}\right\}$, for the $\angle$D is a rt. $\angle$.

Hence $AB^2 = BC^2 + CA^2 - 2BC \cdot CD.$

Q.E.D.

Summary of Theorems 29, 54 and 55.



(i) If the $\angle ACB$ is *obtuse*,

$$AB^2 = BC^2 + CA^2 + 2BC \cdot CD. \qquad \textit{Theor. } 54$$

(ii) If the $\angle ACB$ is *a right angle*,

$$AB^2 = BC^2 + CA^2. \qquad \textit{Theor. } 29.$$

(iii) If the $\angle ACB$ is *acute*,

$$AB^2 = BC^2 + CA^2 - 2BC \cdot CD. \qquad \textit{Theor. } 55.$$

Observe that in (ii), when the $\angle ACB$ is *right*, AD coincides with AC, so that CD (the projection of CA) vanishes;

hence, in this case, $2BC \cdot CD = 0$.

Thus the three results may be collected in a single enunciation:

The square on a side of a triangle is greater than, equal to, or less than the sum of the squares on the other sides, according as the angle contained by those sides is obtuse, a right angle, or acute; the difference in cases of inequality being twice the rectangle contained by one of the two sides and the projection on it of the other.

EXERCISES.

1. In a triangle ABC, $a=21$ cm., $b=17$ cm., $c=10$ cm. By how many square centimetres does c^2 fall short of a^2+b^2? Hence or otherwise calculate the projection of AC on BC.

2. ABC is an isosceles triangle in which AB = AC: and BE is drawn perpendicular to AC. Shew that $BC^2 = 2AC \cdot CE$.

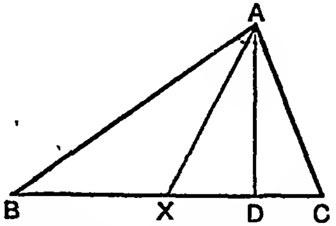
3. In the $\triangle ABC$, shew that

   (i) if the $\angle C = 60°$, then $c^2 = a^2 + b^2 - ab$

   (ii) if the $\angle C = 120°$, then $c^2 = a^2 + b^2 + ab$.

Theorem 56.

In any triangle the sum of the squares on two sides is equal to twice the square on half the third side together with twice the square on the median which bisects the third side.



Let ABC be a triangle, and AX the median which bisects the base BC.

It is required to prove that

$$AB^2 + AC^2 = 2BX^2 + 2AX^2.$$

Draw AD perp. to BC; and consider the case in which AB and AC are unequal, and AD falls within the triangle.

Then of the $\angle$s AXB, AXC, one is obtuse, and the other acute. Let the $\angle$ AXB be obtuse.

Then from the $\triangle$ AXB,

$$AB^2 = BX^2 + AX^2 + 2BX . XD. \qquad \textit{Theor. } 54.$$

And from the $\triangle$ AXC,

$$AC^2 = XC^2 + AX^2 - 2XC . XD. \qquad \textit{Theor. } 55.$$

Adding these results, and remembering that $XC = BX$, we have

$$AB^2 + AC^2 = 2BX^2 + 2AX^2.$$

Q.E.D.

Note. The proof may easily be adapted to the case in which the perpendicular AD falls outside the triangle.

EXERCISE.

In any triangle the difference of the squares on two sides is equal to twice the rectangle contained by the base and the intercept between the middle point of the base and the foot of the perpendicular drawn from the vertical angle to the base.

EXERCISES ON THEOREMS 50–53.

1. Use the Corollaries of Theorem 50 to shew that if a straight line AB is divided internally at X, then

$$AB^2 = AX^2 + XB^2 + 2AX.XB.$$

2. If a straight line AB is bisected at X and produced to Y, and if $AY.YB = 8AX^2$, shew that $AY = 2AB$.

3. *The sum of the squares on two straight lines is never less than twice the rectangle contained by the straight lines.*

Explain this statement by reference to the diagram of Theorem 52.

Also deduce it from the formula $(a-b)^2 = a^2 + b^2 - 2ab$.

4. In the formula $(a+b)(a-b) = a^2 - b^2$, substitute $a = \frac{x+y}{2}$, $b = \frac{x-y}{2}$, and enunciate verbally the resulting theorem.

5. If a straight line is divided internally at Y, shew that the rectangle AY, YB continually diminishes as Y moves from X, the mid-point of AB.

Deduce this (i) from the Corollary of Theorem 53;

(ii) from the formula $ab = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2$.

6. *If a straight line* AB *is bisected at* X, *and also divided* (i) *internally*, (ii) *externally into two unequal segments at* Y, *shew that in either case*

$$AY^2 + YB^2 = 2(AX^2 + XY^2). \qquad \text{[Euclid II. 9, 10.]}$$

[*Proof of case* (i). (30)

$$\begin{aligned} AY^2 + YB^2 &= AB^2 - 2AY.YB & \textit{Theor. } 51. \\ &= 4AX^2 - 2(AX+XY)(AX-XY) \\ &= 4AX^2 - 2(AX^2 - XY^2) & \textit{Theor. } 53. \\ &= 2AX^2 + 2XY^2. \end{aligned}$$

Case (ii) may be derived from Theorem 52 in a similar way.] (31)

7. If AB is divided internally at Y, use the result of the last example to trace the changes in the value of $AY^2 + YB^2$, as Y moves from A to B.

8. *In a right-angled triangle, if a perpendicular is drawn from the right angle to the hypotenuse, the square on this perpendicular is equal to the rectangle contained by the segments of the hypotenuse.*

9. ABC is an isosceles triangle, and AY is drawn to cut the base BC internally or externally at Y. Prove that

$$AY^2 = AC^2 - BY.YC, \text{ for internal section;}$$
$$AY^2 = AC^2 + BY.YC, \text{ for external section.}$$

EXERCISES ON THEOREMS 54–56.

1. AB is a straight line 8 cm. in length, and from its middle point O as centre with radius 5 cm. a circle is drawn; if P is any point on the circumference, shew that

$$AP^2+BP^2=82 \text{ sq. cm.}$$

2. In a triangle ABC, the base BC is bisected at X. If $a=17$ cm., $b=15$ cm., and $c=8$ cm., calculate the length of the median AX, and deduce the $\angle$A.

3. The base of a triangle $=10$ cm., and the sum of the squares on the other sides $=122$ sq. cm.; find the locus of the vertex.

4. Prove that the sum of the squares on the sides of a parallelogram is equal to the sum of the squares on its diagonals.

The sides of a rhombus and its shorter diagonal each measure 3″; find the longer diagonal to within ·01″.

5. In any quadrilateral the squares on the diagonals are together equal to twice the sum of the squares on the straight lines joining the middle points of opposite sides. [See Ex. 7, p. 64.]

6. ABCD is a rectangle, and O any point within it: shew that

$$OA^2+OC^2=OB^2+OD^2.$$

If $AB=6{\cdot}0''$, $BC=2{\cdot}5''$, and $OA^2+OC^2=21\frac{1}{4}$ sq. in., find the distance of O from the intersection of the diagonals.

7. The sum of the squares on the sides of a quadrilateral is greater than the sum of the squares on its diagonals by four times the square on the straight line which joins the middle points of the diagonals.

8. In a triangle ABC, the angles at B and C are acute; if BE, CF are drawn perpendicular to AC, AB respectively, prove that

$$BC^2=AB\,.\,BF+AC\,.\,CE.$$

9. Three times the sum of the squares on the sides of a triangle is equal to four times the sum of the squares on the medians.

10. ABC is a triangle, and O the point of intersection of its medians: shew that

$$AB^2+BC^2+CA^2=3(OA^2+OB^2+OC^2).$$

11. If a straight line AB is bisected at X, and also divided (internally or externally) at Y, then

$$AY^2+YB^2=2(AX^2+XY^2).$$ [See p. 230 Ex. 6.]

Prove this from Theorem 56, by considering a triangle CAB in the limiting position when the vertex C falls at Y in the base AB.

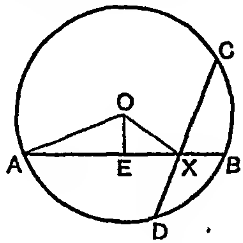
12. In a triangle ABC, if the base BC is divided at X so that mBX$=n$XC, shew that

$$mAB^2+nAC^2=mBX^2+nXC^2+(m+n)AX^2.$$

RECTANGLES IN CONNECTION WITH CIRCLES.

THEOREM 57. [Euclid III. 35.]

If two chords of a circle cut at a point within it, the rectangles contained by their segments are equal.



In the ⊙ ABC, let AB, CD be chords cutting at the internal point X.

It is required to prove that

the rect. AX, XB = *the rect.* CX, XD.

Let O be the centre, and r the radius, of the given circle.

Supposing OE drawn perp. to the chord AB, and therefore bisecting it.

Join OA, OX.

Proof. The rect. AX, XB $= (AE + EX)(EB - EX)$

$= (AE + EX)(AE - EX)$

$= AE^2 - EX^2$ *Theor.* 53.

$= (AE^2 + OE^2) - (EX^2 + OE^2)$

$= \quad r^2 \quad - \quad OX^2$, since the ∠s at E are rt. ∠s.

Similarly it may be shewn that

the rect. CX, XD $= r^2 - OX^2$.

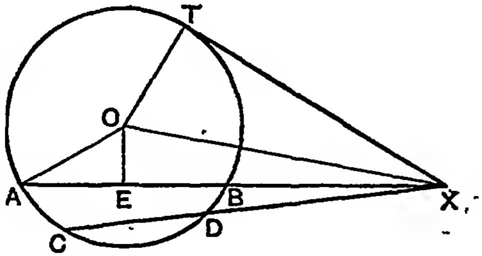
∴ the rect. AX, XB = the rect. CX, XD.

Q.E.D.

COROLLARY. *Each rectangle is equal to the square on half the chord which is bisected at the given point* X.

THEOREM 58. [Euclid III. 36.]

If two chords of a circle, when produced, cut at a point outside it, the rectangles contained by their segments are equal. And each rectangle is equal to the square on the tangent from the point of intersection.



In the ⊙ABC, let AB, CD be chords cutting, when produced, at the external point X; and let XT be a tangent drawn from that point.

It is required to prove that

the rect. AX, XB = *the rect.* CX, XD = *the sq. on* XT.

Let O be the centre, and r the radius of the given circle.

Suppose OE drawn perp. to the chord AB, and therefore bisecting it.

Join OA, OX, OT.

Proof. The rect. AX, XB $= (EX + AE)(EX - EB)$

$= (EX + AE)(EX - AE)$

$= EX^2 - AE^2$ *Theor.* 53.

$= (EX^2 + OE^2) - (AE^2 + OE^2)$

$= OX^2 - r^2$, since the ∠ˢ at E are rt. ∠ˢ.

Similarly it may be shewn that

the rect. CX, XD $= OX^2 - r^2$.

And since the radius OT is perp. to the tangent XT,

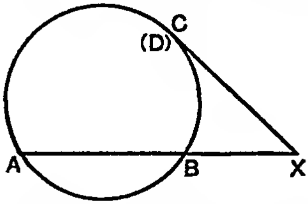
$\therefore\ XT^2 = OX^2 - r^2$. *Theor.* 29.

$\therefore$ the rect. AX, XB = the rect. CX, XD = the sq. on XT.

Q.E.D.

THEOREM 59. [Euclid III. 37.]

If from a point outside a circle two straight lines are drawn, one of which cuts the circle, and the other meets it; and if the rectangle contained by the whole line which cuts the circle and the part of it outside the circle is equal to the square on the line which meets the circle, then the line which meets the circle is a tangent to it.



From X a point outside the ⊙ABC, let two straight lines XA, XC be drawn, of which XA cuts the circle at A and B, and XC meets it at C;

and let the rect. XA, XB = the sq. on XC.

It is required to prove that XC *touches the circle at* C.

Proof. Suppose XC meets the circle again at D;

then $XA . XB = XC . XD$. *Theor.* 58.

But by hypothesis, $XA . XB = XC^2$;

$\therefore XC . XD = XC^2$;

$\therefore XD = XC$.

Hence XC cannot meet the circle again unless the points of section coincide;

that is, XC is a tangent to the circle.

Q.E.D.

NOTE ON THEOREMS 57, 58.

Remembering that the segments into which the chord AB is divided at X, internally in Theorem 57, and externally in Theorem 58, are in each case AX, XB, we may include both Theorems in a single enunciation.

If any number of chords of a circle are drawn through a given point within or without a circle, the rectangles contained by the segments of the chords are equal.

EXERCISES ON THEOREMS 57–59.

(*Numerical and Graphical.*)

1. Draw a circle of radius 5 cm., and within it take a point X 3 cm. from the centre O. Through X draw any two chords AB, CD.

(i) Measure the segments of AB and CD; hence find approximately the areas of the rectangles AX.XB and CX.XD, and compare the results.

(ii) Draw the chord MN which is bisected at X; and from the right-angled triangle OXM calculate the value of XM^2.

(iii) Find by how much per cent. your estimate of the rect. AX, XB differs from its true value.

2. Draw a circle of radius 3 cm., and take an external point X 5 cm. from the centre O. Through X draw any two secants XAB, XCD.

(i) Measure XA, XB and XC, XD; hence find approximately the rectangles XA.XB and XC.XD, and compare the results.

(ii) Draw the tangent XT; and from the right-angled triangle XTO calculate the value of XT^2.

(iii) Find by how much per cent. your estimate of the rect. AX, XB differs from its true value.

3. AB, CD are two straight lines intersecting at X. AX=1·8″, XB=1·2″, and CX=2·7″. If A, C, B, D are concyclic, find the length of XD.

Draw a circle through A, C, B, and check your result by measurement.

4. A secant XAB and a tangent XT are drawn to a circle from an external point X.

(i) If XA=0·6″, and XB=2·4″, find XT.
(ii) If XT=7·5 cm., and XA=4·5 cm., find XB.

5. A semi-circle is drawn on a given line AB; and from X, any point in AB, a perpendicular XM is drawn to AB cutting the circumference at M: shew that

$$AX.XB=MX^2.$$

(i) If AX=2·5″, and MX=2·0″, find XB; hence find the diameter of the semi-circle.

(ii) If the radius of the semi-circle=3·7 cm., and AX=4·9 cm., find MX.

6. A point X moves within a circle of radius 4 cm., and PQ is any chord passing through X; if in all positions PX.XQ=12 sq. cm., find the locus of X.

What will the locus be if X moves outside the same circle, so that PX.XQ=20 sq. cm.?

EXERCISES ON THEOREMS 57–59.

(Theoretical.)

1. ABC is a triangle right-angled at C; and from C a perpendicular CD is drawn to the hypotenuse: shew that

$$AD.DB = CD^2.$$

2. If two circles intersect, and through any point X in their common chord two chords AB, CD are drawn, one in each circle, shew that

$$AX.XB = CX.XD.$$

3. Deduce from Theorem 58 that the tangents drawn to a circle from any external point are equal.

4. If two circles intersect, tangents drawn to them from any point in their common chord produced are equal.

5. If a common tangent PQ is drawn to two circles which cut at A and B, show that AB produced bisects PQ.

6. If two straight lines AB, CD intersect at X so that $AX.XB = CX.XD$, deduce from Theorem 57 (by *reductio ad absurdum*) that the points A, B, C, D are concyclic.

7. In the triangle ABC, perpendiculars AP, BQ are drawn from A and B to the opposite sides, and intersect at O: shew that

$$AO.OP = BO.OQ.$$

8. ABC is a triangle right-angled at C, and from C a perpendicular CD is drawn to the hypotenuse: shew that

$$AB.AD = AC^2.$$

9. Through A, a point of intersection of two circles, two straight lines CAE, DAF are drawn, each passing through a centre and terminated by the circumferences: shew that

$$CA.AE = DA.AF.$$

10. If from any external point P two tangents are drawn to a given circle whose centre is O and radius r; and if OP meets the chord of contact at Q; shew that

$$OP.OQ = r^2.$$

11. AB is a fixed diameter of a circle, and CD is perpendicular to AB (or AB produced); if *any* straight line is drawn from A to cut CD at P and the circle at Q, shew that

$$AP.AQ = \text{constant}.$$

12. A is a fixed point, and CD a fixed straight line; AP is *any* straight line drawn from A to meet CD at P; if in AP a point Q is taken so that $AP.AQ$ is constant, find the locus of Q.

EXERCISES ON THEOREMS 57–59.

(*Miscellaneous.*)

. The chord of an arc of a circle $=2c$, the height of the arc $=h$, the radius $=r$. Shew by Theorem 57 that

$$h(2r-h)=c^2.$$

Hence find the diameter of a circle in which a chord 24″ long cuts off a segment 8″ in height.

2. The radius of a circular arch is 25 feet, and its height is 18 feet: find the span of the arch.

If the height is reduced by 8 feet, the radius remaining the same, by how much will the span be reduced?

Check your calculated results graphically by a diagram in which 1″ represents 10 feet.

3. Employ the equation $h(2r-h)=c^2$ to find the height of an arc whose chord is 16 cm., and radius 17 cm.

Explain the double result geometrically.

4. If d denotes the shortest distance from an external point to a circle, and t the length of the tangent from the same point, shew by Theorem 58 that

$$d(d+2r)=t^2.$$

Hence find the diameter of the circle when $d=1{\cdot}2''$, and $t=2{\cdot}4''$; and verify your result graphically.

5. If the horizon visible to an observer on a cliff 330 feet above the sea-level is $22\frac{1}{4}$ miles distant, find roughly the diameter of the earth.

Hence find the approximate distance at which a bright light raised 66 feet above the sea is visible at the sea-level.

6. If h is the height of an arc of radius r, and b the chord of half the arc, prove that

$$b^2=2rh.$$

7. A semi-circle is described on AB as diameter, and any two chords AC, BD are drawn intersecting at P: shew that

$$AB^2=AC.AP+BD.BP.$$

8. Two circles intersect at B and C, and the two direct common tangents AE and DF are drawn: if the common chord is produced to meet the tangents at G and H, shew that

$$GH^2=AE^2+BC^2.$$

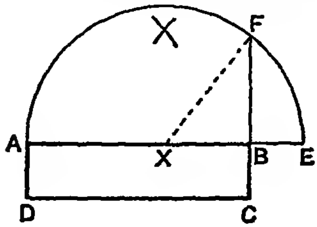
9. If from an external point P a secant PCD is drawn to a circle, and PM is perpendicular to a diameter AB, shew that

$$PM^2=PC.PD+AM.MB.$$

PROBLEMS.

PROBLEM 32.

To draw a square equal in area to a given rectangle.



X

Let ABCD be the given rectangle.

Construction. Produce AB to E, making BE equal to BC. On AE draw a semi-circle; and produce CB to meet the circumference at F.

Then BF is a side of the required square.

Proof. Let X be the mid-point of AE, and r the radius of the semi-circle. Join XF.

Then the rect. $AC = AB \cdot BE$

$= (r + XB)(r - XB)$

$= r^2 - XB^2$

$= FB^2$, from the rt. angled $\triangle$ FBX.

COROLLARY. *To describe a square equal in area to any given rectilineal figure.*

Reduce the given figure to a triangle of equal area. *Prob.* 18.
Draw a rectangle equivalent to this triangle. *Prob.* 17.
Apply to the rectangle the construction given above.

EXERCISES.

1. Draw a rectangle 8 cm. by 2 cm., and construct a square of equal area. What is the length of each side?

2. Find graphically the side of a square equal in area to a rectangle whose length and breadth are 3·0″ and 1·5″. Test your work by measurement and calculation.

3. Draw any rectangle whose area is 3·75 sq. in.; and construct a square of equal area. Find by measurement and calculation the length of each side.

4. Draw an equilateral triangle on a side of 3″, and construct a rectangle of equal area [Problem 17]. Hence find by construction and measurement the side of an equal square.

5. Draw a quadrilateral ABCD from the following data: A=65°; AB=AD=9 cm.; BC=CD=5 cm. Reduce this figure to a triangle [Problem 18], and hence to a rectangle of equal area. Construct an equal square, and measure the length of its side.

6. Divide AB, a line 9 cm. in length, internally at X, so that AX.XB=the square on a side of 4 cm.

Hence give a graphical solution, correct to the first decimal place, of the simultaneous equations:

$$x+y=9, \qquad xy=16.$$

7. Taking $\frac{1}{10}''$ as your unit of length, solve the following equations by a graphical construction, correct to one decimal figure:

$$x+y=40, \qquad xy=169.$$

8. The area of a rectangle is 25 sq. cm., and the length of one side is 7·2 cm.; find graphically the length of the other side to the nearest millimetre, and test your drawing by calculation.

9. Divide AB, a line 8 cm. in length, *externally* at X, so that AX.XB=the square on a side of 6 cm. [See p. 245.]

Hence find a graphical solution, correct to the first decimal place, of the equations:

$$x-y=8, \qquad xy=36.$$

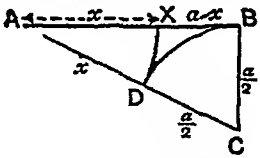
10. On a straight line AB draw a semi-circle, and from any point P on the circumference draw PX perpendicular to AB. Join AP, PB, and denote these lines by x and y.

Noticing that (i) $x^2+y^2=AB^2$; (ii) $xy=2\triangle APB=AB.PX$; devise a graphical solution of the equations:

$$x^2+y^2=100; \qquad xy=25.$$

PROBLEM 33.

To divide a given straight line so that the rectangle contained by the whole and one part may be equal to the square on the other part.



Let AB be the st. line to be divided at a point X in such a way that

$$AB \,.\, BX = AX^2.$$

Construction. Draw BC perp. to AB, and make BC equal to half AB. Join AC.

From CA cut off CD equal to CB.

From AB cut off AX equal to AD.

Then AB is divided as required at X.

Proof. Let $AB = a$ units of length, and let $AX = x$.

Then $BX = a - x$; $AD = x$; $BC = CD = \dfrac{a}{2}$.

Now $AB^2 = AC^2 - BC^2$, from the rt. angled $\triangle ABC$;

$= (AC - BC)(AC + BC)$;

that is, $a^2 = x(x + a)$

$= x^2 + ax$.

From each of these equals take ax;

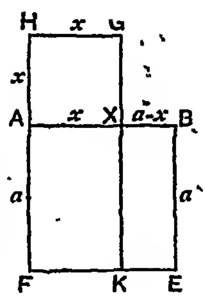
then $a^2 - ax = x^2$;

or, $a(a - x) = x^2$,

that is, $AB \,.\, BX = AX^2$.

EXERCISE.

Let AB be divided as above at X. On AB, AX, and on opposite sides of AB, draw the squares ABEF, AXGH; and produce GX to meet FE at K. In this diagram name rectangular figures equivalent to a^2, x^2, $x(x+a)$, ax, and $a(a-x)$. Hence illustrate the above proof graphically.

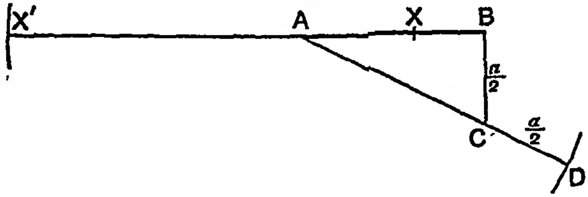


NOTE. A straight line is said to be divided in Medial Section when the rectangle contained by the given line and one segment is equal to the square on the other segment.

This division may be *internal* or *external*; that is to say, AB may be divided internally at X, and externally at X′, so that

(i) $AB . BX = AX^2$,

(ii) $AB . BX' = AX'^2$.



To obtain X′, the construction of p. 240 must be modified thus:

CD is to be cut off from AC *produced*;

AX′..................... from BA *produced*, in the *negative* sense.

ALGEBRAICAL ILLUSTRATION.

If a st. line AB is divided at X, internally or externally, so that

$$AB . BX = AX^2,$$

and if $AB = a$, $AX = x$, and consequently $BX = a - x$, then

$$a(a - x) = x^2,$$

or, $$x^2 + ax - a^2 = 0,$$

and the roots of this quadratic, namely, $\frac{a\sqrt{5}}{2} - \frac{a}{2}$ and $-\left(\frac{a\sqrt{5}}{2} + \frac{a}{2}\right)$, are the lengths of AX and AX′.

EXERCISES.

1. Divide a straight line 4″ long internally in medial section. Measure the greater segment, and find its length algebraically.

2. Divide AB, a line 2″ long, externally in medial section at X′. Measure AX′, and obtain its length algebraically, explaining the geometrical meaning of the negative sign.

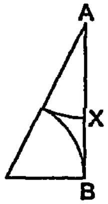
3. In the figure of Problem 33, shew that $AC = \frac{a\sqrt{5}}{2}$. [Theor. 29.]

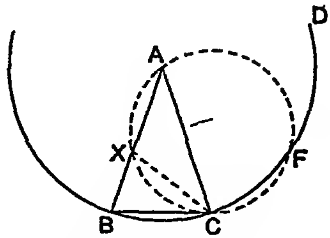
Hence prove (i) $AX = \frac{a\sqrt{5}}{2} - \frac{a}{2}$; (ii) $AX' = -\left(\frac{a\sqrt{5}}{2} + \frac{a}{2}\right)$.

4. If a straight line is divided internally in medial section, and from the greater segment a part is taken equal to the less, shew that the greater segment is also divided in medial section.

Problem 34.

To draw an isosceles triangle having each of the angles at the base double of the vertical angle.





Construction. Take any line AB, and divide it at X, so that $AB \,.\, BX = AX^2$. *Prob.* 33.

(This construction is shewn separately on the left.)

With centre A, and radius AB, draw the $\odot$ BCD; and in it place the chord BC equal to AX.

Join AC.

Then ABC is the triangle required.

Proof. Join XC, and suppose a circle drawn through A, X and C.

Now, by construction, $BA \,.\, BX = AX^2$
$= BC^2$;

$\therefore$ BC touches the $\odot$ AXC at C; *Theor.* 59.

$\therefore$ the $\angle$ BCX = the $\angle$ XAC, in the alt. segment.

To each add the $\angle$ XCA;

then the $\angle$ BCA = the $\angle$ XAC + the $\angle$ XCA
= the ext. $\angle$ CXB.

And the $\angle$ BCA = the $\angle$ CBA, for AB = AC.

$\therefore$ the $\angle$ CBX = the $\angle$ CXB;

$\therefore$ CX = CB = AX,

$\therefore$ the $\angle$ XAC = the $\angle$ XCA;

$\therefore$ the $\angle$ XAC + the $\angle$ XCA = twice the $\angle$ A.

But the $\angle$ ABC = the $\angle$ ACB = the $\angle$ XAC + the $\angle$ XCA *Proved.*
= twice the $\angle$ A.

EXERCISES.

1. How many degrees are there in the vertical angle of an isosceles triangle in which each angle at the base is double of the vertical angle?

2. Shew how a right angle may be divided into five equal parts by means of Problem 34.

3. In the figure of Problem 34 point out a triangle whose vertical angle is three times either angle at the base.

Shew how such a triangle may be constructed.

4. If in the triangle ABC, the $\angle B$=the $\angle C$=twice the $\angle A$, shew that

$$\frac{BC}{AB}=\frac{\sqrt{5}-1}{2}.$$

5. In the figure of Problem 34, if the two circles intersect at F, shew that

(i) BC=CF;

(ii) the circle AXC=the circum-circle of the triangle ABC;

(iii) BC, CF are sides of a regular decagon inscribed in the circle BCD;

(iv) AX, XC, CF are sides of a regular pentagon inscribed in the circle AXC.

6. In the figure of Problem 34, shew that the centre of the circle circumscribed about the triangle CBX is the middle point of the arc XC.

7. In the figure of Problem 34, if I is the in-centre of the triangle ABC, and I′, S′ the in-centre and circum-centre of the triangle CBX, shew that S′I=S′I′.

8. If a straight line is divided in medial section, the rectangle contained by the sum and difference of the segments is equal to the rectangle contained by the segments.

9. If a straight line AB is divided internally in medial section at X, shew that

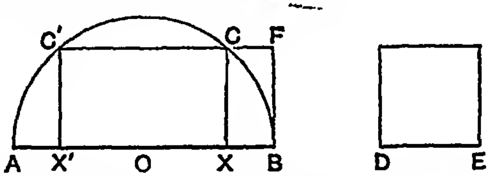
$$AB^2+BX^2=3AX^2.$$

Also verify this result by substituting the values given on page 241.

THE GRAPHICAL SOLUTION OF QUADRATIC EQUATIONS.

From the following constructions, which depend on Problem 32, a graphical solution of easy quadratic equations may be obtained.

I. *To divide a straight line* **internally** *so that the rectangle contained by the segments may be equal to a given square.*



Let AB be the st. line to be divided, and DE a side of the given square.

Construction. On AB draw a semicircle; and from B draw BF perp. to AB and equal to DE.

From F draw FCC' par^l to AB, cutting the O^{ce} at C and C'.

From C, C' draw CX, C'X' perp. to AB.

Then AB is divided as required at X, and also at X'.

Proof. $AX \cdot XB = CX^2$ *Prob.* 32.

$= BF^2$

$= DE^2$.

Similarly $AX' \cdot X'B = DE^2$.

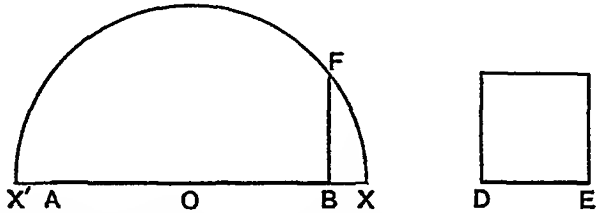
Application. The purpose of this construction is to find two straight lines AX, XB, having given their *sum*, viz. AB, and their *product*, viz. the square on DE.

Now to solve the equation $x^2 - 13x + 36 = 0$, we have to find two numbers whose *sum* is 13, and whose *product* is 36, or 6^2.

To do this graphically, perform the above construction, making AB equal to 13 cm., and DE equal to $\sqrt{36}$ or 6 cm. The segments AX, XB represent the roots of the equation, and their values may be obtained by measurement.

Note. If the last term of the equation is not a perfect square, as in $x^2 - 7x + 11 = 0$, $\sqrt{11}$ must be first got by the arithmetical rule, or graphically by means of Problem 32.

VII. *To divide a straight line* externally *so that the rectangle contained by the segments may be equal to a given square.*



Let AB be the st. line to be divided externally, and DE the side of the given square.

Construction. From B draw BF perp. to AB, and equal to DE. Bisect AB at O.

With centre O, and radius OF draw a semi-circle to cut AB produced at X and X′.

Then AB is divided externally as required at X, and also at X′.

Proof. $$AX \cdot XB = X'B \cdot BX, \text{ since } AX = X'B,$$
$$= BF^2 \qquad \textit{Prob. } 32.$$
$$= DE^2.$$

Application. Here we find two lines AX, XB, having given their *difference*, viz. AB, and their *product*, viz. the square on DE.

Now to solve the equation $x^2-6x-16=0$, we have to find two numbers whose numerical *difference* is 6, and whose product is 16, or 4^2.

To do this graphically, perform the above construction, making AB equal to 6 cm., and DE equal to $\sqrt{16}$ or 4 cm. The segments AX, XB represent the roots of the equation, and their values, as before, may be obtained by measurement.

EXERCISES.

Obtain approximately the roots of the following quadratics by means of graphical constructions; and test your results algebraically.

1. $x^2-10x+16=0.$ 2. $x^2-14x+49=0.$ 3. $x^2-12x+25=0.$

4. $x^2-5x-36=0.$ 5. $x^2-7x-49=0.$ 6. $x^2-10x+20=0.$

EXERCISES FOR SQUARED PAPER.

1. A circle passing through the points (0, 4), (0, 9) touches the x-axis at P. Calculate and measure the length of OP.

2. With centre at the point (9, 6) a circle is drawn to touch the y-axis. Find the rectangle of the segments of any chord through O. Also find the rectangle of the segments of any chord through the point (9, 12).

3. Draw a circle (shewing all lines of construction) through the points (6, 0), (24, 0), (0, 9). Find the length of the other intercept on the y-axis, and verify by measurement. Also find the length of a tangent to the circle from the origin.

4. Draw a circle through the points (10, 0), (0, 5), (0, 20); and prove by Theorem 59 that it touches the x-axis.

Find (i) the coordinates of the centre, (ii) the length of the radius.

5. If a circle passes through the points (16, 0), (18, 0), (0, 12), show by Theorem 58 that it also passes through (0, 24).

Find (i) the coordinates of the centre, (ii) the length of the tangent from the origin.

6. Plot the points A, B, C, D from the coordinates (12, 0), (−6, 0), (0, 9), (0, −8); and prove by Theorem 57 that they are concyclic.

If r denotes the radius of the circle, shew that

$$OA^2+OB^2+OC^2+OD^2=4r^2.$$

7. Draw a circle (shewing all lines of construction) to touch the y-axis at the point (0, 9), and to cut the x-axis at (3, 0).

Prove that the circle must cut the x-axis again at the point (27, 0); and find its radius. Verify your results by measurement.

8. Show that two circles of radius 13 may be drawn through the point (0, 8) to touch the x-axis; and by means of Theorem 58 find the length of their common chord.

9. Given a circle of radius 15, the centre being at the origin, show how to draw a second circle of the same radius touching the given circle and also touching the x-axis.

How many circles can be so drawn? Measure the coordinates of the centre of that in the first quadrant.

10. A, B, C, D are four points on the x-axis at distances 6, 9, 15, 25 from the origin O. Draw two intersecting circles, one through A, B, and the other through C, D, and hence determine a point P in the x-axis such that

$$PA.PB=PC.PD.$$

Calculate and measure OP.

If the distances of A, B, C, D from O are a, b, c, d respectively, prove that

$$OP=(ab-cd)/(a+b-c-d).$$

PART V.

ON PROPORTION.

DEFINITIONS AND FIRST PRINCIPLES.

The ratio of one magnitude to another *of the same kind* is the relation which the first bears to the second in regard to quantity; this is measured by the fraction which the first is of the second.

Thus if two such magnitudes contain a and b units respectively, the ratio of the first to the second is expressed by the fraction $\frac{a}{b}$.

The ratio of a to b is generally denoted thus, $a : b$; and a is called the antecedent and b the consequent of the ratio.

The two magnitudes compared in a ratio must be *of the same kind*; for example, both must be lines, or both angles, or both areas. It is clearly impossible to compare the *length* of a straight line with a magnitude of a different kind, such as the *area* of a triangle. Moreover, a ratio is an *abstract* fraction. Thus the ratio which a line 6 cm. long bears to a line 8 cm. long is $\frac{6}{8}$ or $\frac{3}{4}$, (not $\frac{3}{4}$ *cm.*).

NOTE. It is not always possible to express two quantities of the same kind in terms of a common unit. For instance, if the side of a square is 1 inch, the diagonal is $\sqrt{2}$ inches. But since the numerical value of $\sqrt{2}$ cannot be *exactly* determined (though it can be found to any number of decimal figures), the side and diagonal cannot be expressed in terms of the same unit. Two such quantities are said to be incommensurable. But by choosing a sufficiently small quantity as unit, two incommensurables, such as $\sqrt{2}$ inches and 1 inch, may be expressed to any required degree of accuracy. Thus, remembering that $\sqrt{2} = 1{\cdot}41421\ldots$, it follows that $\sqrt{2}$ inches and 1 inch may be represented by

1414 and 1000, roughly, taking $\frac{1}{1000}''$ as unit,

14142 and 10000, more nearly, taking $\frac{1}{10000}''$ as unit;

and so on.

2. If a point X is taken in a given line AB, or in AB produced, the ratio in which it divides AB is the ratio of the *segments* of AB, namely AX : XB, whether the division is *internal* as in Fig. 1, or *external* as in Fig. 2.

Fig. 1.

Fig. 2.

3. Four magnitudes are in proportion, when the ratio of the *first* to the *second* is equal to the ratio of the *third* to the *fourth*.

When the ratio a to b is equal to that of x to y, the four magnitudes are called proportionals.

This is expressed by saying "*a is to b as x is to y*"; and the proportion is written

$$\frac{a}{b}=\frac{x}{y},$$

or $$a:b=x:y.$$

Here a and y are called the extremes, and b and x the means; and y is said to be a fourth proportional to a, b, and x.

In a proportion, terms which are both antecedents or both consequents of the ratios are said to be corresponding terms.

NOTE. In a proportion such as $a:b=x:y$, the magnitudes compared in *each ratio* must be of the same kind, though the magnitudes of the second ratio need not be of the same kind as those of the first. For instance, a and b may denote *areas*, and x and y *lines*; in which case the proportion asserts that the first area bears the same ratio to the second area, as the first line bears to the second line.

4. Three magnitudes of the same kind are said to be proportionals, when the ratio of the *first* to the *second* is equal to that of the *second* to the *third*.

Thus a, b, c are proportionals if

$$\frac{a}{b}=\frac{b}{c};$$

or $$a:b=b:c.$$

Here b is called a mean proportional between a and c;
and c is called a third proportional to a and b.

AXIOMS.

(i) *Ratios which are equal to the same ratio are equal to one another.*

For instance, if $a:b=x:y$, and $c:d=x:y$,
then evidently $a:b=c:d$.

(ii) *Magnitudes which bear the same ratio to the same magnitude are equal to one another.*

For instance, if $a : x = b : x$,

then evidently $a = b$.

Introductory Theorems.

I. *If four magnitudes are proportionals, they are also proportionals when taken* **inversely.**

That is, if $a : b = x : y$,

then $b : a = y : x$.

For, by hypothesis, $\frac{a}{b} = \frac{x}{y}$; hence $\frac{b}{a} = \frac{y}{x}$;

or $b : a = y : x$.

II. *If four magnitudes of the same kind are proportionals, they are also proportionals when taken* **alternately.**

That is, if $a : b = x : y$,

then $a : x = b : y$.

For, by hypothesis, $\frac{a}{b} = \frac{x}{y}$;

multiplying both sides by $\frac{b}{x}$,

we have $\frac{a}{b} \cdot \frac{b}{x} = \frac{x}{y} \cdot \frac{b}{x}$;

that is, $\frac{a}{x} = \frac{b}{y}$,

or $a : x = b : y$.

Note. In this theorem the *hypothesis* requires that a and b shall be of the same kind, also that x and y shall be of the same kind; while the *conclusion* requires that a and x shall be of the same kind, and also b and y of the same kind.

III. *If four numbers are proportional, the* product of the extremes *is equal to the* product of the means.

That is, if $a : b = c : d$,

then $ad = bc$.

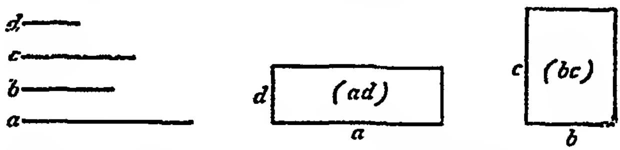
For, by hypothesis, $\frac{a}{b} = \frac{c}{d}$;

multiplying each side of this equation by bd, we have

$$ad = bc.$$

COROLLARY. If a, b, c, d denote the lengths of four straight lines in proportion, the above result states that *the* rectangle *contained by the extremes is equal to the* rectangle *contained by the means.*

This is illustrated by the following diagram:



Similarly if three lines a, b, c are proportionals,

that is, if $a : b = b : c$;

then $ac = b^2$.

Or, *the* rectangle *contained by the extremes is equal to the* square *on the mean.*

IV. *If there are four magnitudes in proportion, the* sum *(or* difference*) of the first and second is to the second as the* sum *(or* difference*) of the third and fourth is to the fourth.*

That is, if $a : b = x : y$;

then (i) $a + b : b = x + y : y$;

(ii) $a - b : b = x - y : y$.

For by hypothesis, $$\frac{a}{b}=\frac{x}{y};$$

$$\therefore \frac{a}{b}+1=\frac{x}{y}+1, \text{ or } \frac{a+b}{b}=\frac{x+y}{y};$$

that is, $$a+b:b=x+y:y. \quad\text{(i)}$$

This inference is sometimes referred to as componendo.

Similarly by subtracting 1 from the equal ratios $\frac{a}{b}$, $\frac{x}{y}$, we obtain

$$\frac{a-b}{b}=\frac{x-y}{y};$$

that is, $$a-b:b=x-y:y. \quad\text{(ii)}$$

This inference is sometimes referred to as dividendo.

COROLLARY. If $a:b=x:y$,

then $$a+b:a-b=x+y:x-y.$$

This is obtained by dividing the result of (i) by that of (ii).

In a series of equal ratios (the magnitudes being all of the same kind), as any antecedent is to its consequent so is the sum of the antecedents to the sum of the consequents.

That is, if $$\frac{a}{x}=\frac{b}{y}=\frac{c}{z}=\ldots;$$

then $$\frac{a}{x}=\frac{a+b+c+\ldots}{x+y+z+\ldots}.$$

Let each of the equal ratios $\frac{a}{x}$, $\frac{b}{y}$, $\frac{c}{z}$, ... be equal to k.

Then $$a=kx,\ b=ky,\ c=kz,\ \ldots;$$

$\therefore$, by addition,

$$a+b+c+\ldots=k(x+y+z+\ldots);$$

$$\therefore \frac{a+b+c+\ldots}{x+y+z+\ldots}=k=\frac{a}{x},$$

or $$a:x=a+b+c+\ldots:x+y+z+\ldots.$$

VI. *A given straight line can be divided internally in a given ratio at one, and only one, point; and externally at one, and only one, point.*

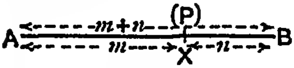


Fig. 1.

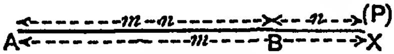


Fig. 2.

Let AB be the given line, and $m : n$ the given ratio, m being greater than n.

Internal Division. (i) Divide AB (Fig. 1) into $m+n$ equal parts [*Prob.* 7]; and of these parts make AX to contain m; then XB must contain n.

Hence $$\text{AX} : \text{XB} = m : n;$$

that is, AB is divided *internally* at X in the given ratio.

(ii) Again, since AX and AB contain respectively m and $m+n$ equal parts,

$$\therefore\ \text{AX} : \text{AB} = m : m+n.$$

Similarly, if P divides AB in the given ratio $m : n$,

$$\text{AP} : \text{AB} = m : m+n.$$

$$\therefore\ \frac{\text{AX}}{\text{AB}} = \frac{\text{AP}}{\text{AB}};$$

$$\therefore\ \text{AX} = \text{AP}.$$

Hence P and X coincide; that is, X is the only point which divides AB internally in the ratio $m : n$.

External Division. (i) Divide AB (Fig. 2) into $m-n$ equal parts; and in AB produced make AX to contain m such parts; then XB must contain n.

Hence $$\text{AX} : \text{XB} = m : n;$$

that is AB is divided *externally* at X in the given ratio.

(ii) And it may be shewn, as above, that X is the only point which divides AB externally in the ratio $m : n$.

EXERCISES.

1. Insert the missing terms in the following proportions:

(i) $3:7=15:(\quad)$;

(ii) $2{\cdot}5:(\quad)=10:32$;

(iii) $(\quad):ac^2=bc:bc^3$.

2. Correct the following statement:

$$£65:78 \text{ ft.}=£25:30 \text{ ft.}$$

3. If a straight line, 9·6″ in length, is divided *internally* in the ratio 5 : 7, calculate the lengths of the segments.

4. If a straight line 4·5 cm. in length is divided *externally* in the ratio 11 : 8, calculate the lengths of the segments.

5. AB is a straight line, 6·4 cm. in length, divided *internally* at X and *externally* at Y in the ratio 5 : 3; calculate the lengths of the segments, and shew that they satisfy the formula

$$\frac{2}{AB}=\frac{1}{AX}+\frac{1}{AY}.$$

6. If a straight line, a inches in length, is divided *internally* in the ratio $m:n$, shew that the lengths of the segments are respectively

$$\frac{m}{m+n}.a \text{ inches,} \quad \frac{n}{m+n}.a \text{ inches.}$$

7. If a straight line, a units in length, is divided *externally* in the ratio $m:n$, shew that the lengths of the segments are respectively

$$\frac{m}{m-n}.a \text{ units,} \quad \frac{n}{m-n}.a \text{ units.}$$

8. If $a:b=x:y$, and $b:c=y:z$, prove that $a:c=x:z$.

9. If $a:b=x:y$, shew that $a+b:a=x+y:x$.

10. If a, b, c are three proportionals, shew that $a:c=a^2:b^2$.

11. If two straight lines AB, CD are divided internally in the same ratio at X and Y respectively, shew that

(i) AB : XB = CD : YD;

(ii) AB : AX = CD : CY.

12. If a, b, c, d are four straight lines such that the rectangle contained by a and d is equal to that contained by b and c, prove that

$$a:b=c:d.$$

PROPORTIONAL DIVISION OF STRAIGHT LINES.

THEOREM 60. [Euclid VI. 2.]

A straight line drawn parallel to one side of a triangle cuts the other two sides, or those sides produced, proportionally.

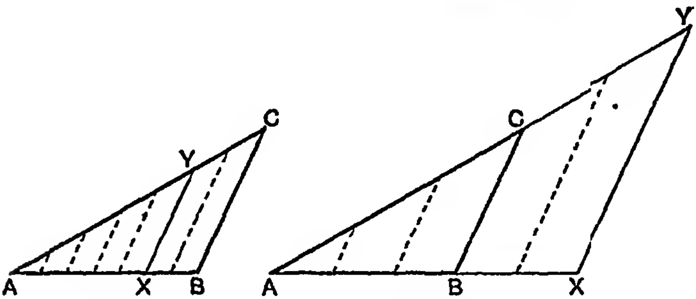


Fig. 1. Fig. 2.

In the $\triangle ABC$, let XY, drawn par[1] to the side BC, cut AB, AC at X and Y, internally in Fig. 1, externally in Fig. 2.

It is required to prove in both cases that

$$AX : XB = AY : YC.$$

Proof. Suppose X divides AB in the ratio $m : n$; that is, suppose

$$AX : XB = m : n;$$

so that, if AX is divided into m equal parts, then XB may be divided into n such equal parts.

Through the points of division in AX, XB let parallels be drawn to BC.

Then these parallels divide the segments AY, YC into parts which are all equal; *Theor.* 22.

and of these equal parts AY contains m,

and YC contains n;

hence $AY : YC = m : n.$

$$\therefore AX : XB = AY : YC$$

Q.E.D.

Conversely, *if a line cuts two sides of a triangle proportionally, it is parallel to the third side.*

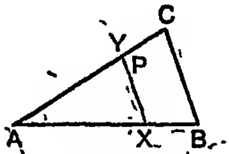


Fig. 1.

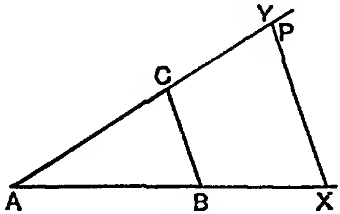


Fig. 2.

Conversely, let XY cut the sides AB, AC proportionally, so that

$$AX : XB = AY : YC.$$

It is required to prove that XY *is parallel to* BC.

Let XP be drawn through X par[1] to BC, to meet AC in P.

Then $\quad AP : PC = AX : XB;$

but, by hypothesis, $\quad AY : YC = AX : XB.$

Thus AC is cut, internally in Fig. 1, and externally in Fig. 2 in the same ratio at P and Y.

Hence P coincides with Y, and consequently XP with XY.

Theor. VI. p. 252

That is, $\quad$ XY is par[1] to BC.

Q.E.D.

Corollary. If XY is parallel to BC, then

$$AX : AB = AY : AC.$$

For, taking Fig. 1, it may be shewn that

$$AX : AB = m : m+n;$$

and hence, by Theorem 22, that

$$AY : AC = m : m+n.$$

$$\therefore\ AX : AB = AY : AC.$$

Conversely, if $\quad AX : AB = AY : AC.$

it may be proved as above that XY is par[1] to BC.

THEOREM 61. [Euclid VI. 3 and A.]

If the vertical angle of a triangle is bisected internally or externally, the bisector divides the base internally or externally into segments which have the same ratio as the other sides of the triangle.

Conversely, if the base is divided internally or externally into segments proportional to the other sides of the triangle, the line joining the point of section to the vertex bisects the vertical angle internally or externally.

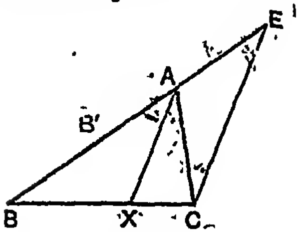


Fig. 1.

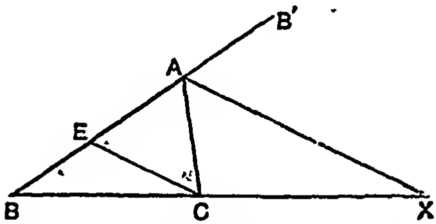


Fig. 2.

In the $\triangle$ ABC, let AX bisect the $\angle$ BAC, internally in Fig. 1, and externally in Fig. 2; that is, in the latter case, let AX bisect the exterior $\angle$ B′AC.

It is required to prove in both cases that

$$BX : XC = BA : AC.$$

Let CE be drawn through C par[l] to XA to meet BA (produced if necessary) at E. In Fig. 1 let a point B′ be taken in AB.

Proof. Because XA and CE are par[l],

$\therefore$, in both Figs., the $\angle$ B′AX = the int. opp. $\angle$ AEC

Also, by hypothesis,

$$\text{the } \angle B'AX = \text{the } \angle XAC$$
$$= \text{the alt. } \angle ACE.$$
$$\therefore \text{ the } \angle AEC = \text{the } \angle ACE:$$
$$\therefore AC = AE.$$

Again, because XA is par[l] to CE, a side of the $\triangle$ BCE.

$\therefore$, in both Figs., $BX : XC = BA : AE$;

that is, $BX : XC = BA : AC$.

Q.E.D.

Conversely, let BC be divided internally (Fig. 1) or externally (Fig. 2) at X, so that $BX : XC = BA : AC$.

It is required to prove that the $\angle B'AX =$ the $\angle XAC$.

Proof. For, with the same construction as before,

because XA is par[l] to CE, a side of the $\triangle BCE$,

$\therefore BX : XC = BA : AE$.

But, by hypothesis, $BX : XC = BA : AC$;

$\therefore BA : AC = BA : AE$;

$\therefore AC = AE$.

$\therefore$ the $\angle AEC =$ the $\angle ACE$

$=$ the alt. $\angle XAC$.

And in both Figs.,

the ext. $\angle B'AX =$ the int. opp. $\angle AEC$;

$\therefore$ the $\angle B'AX =$ the $\angle XAC$.

Q.E.D.

DEFINITION.

When a finite straight line is divided internally and externally into segments which have the same ratio, it is said to be cut harmonically.

Hence the following Corollary to Theorem 61.

The base of a triangle is divided harmonically by the internal and external bisectors of the vertical angle:

for in each case the segments of the base are in the ratio of the other sides of the triangle.

[For Theorems and Examples on Harmonic Section see p. 323.]

H.S.G

EXERCISES ON THEOREM 60.

(Numerical and Graphical.)

1. On a base AB, 3·5″ in length, draw any triangle CAB; and from AB cut off AX 2·1″ long. Through X draw XY parallel to BC to meet AC at Y.

Measure AY, YC; and hence compare the ratios

$$\text{(i)}\ \frac{AX}{XB}, \frac{AY}{YC}; \quad \text{(ii)}\ \frac{AB}{AX}, \frac{AC}{AY}; \quad \text{(iii)}\ \frac{AB}{XB}, \frac{AC}{YC}.$$

2. ABC is a triangle, and XY is drawn parallel to BC, cutting the other sides at X and Y.

(i) If AB=3·6″, AC=2·4″, and AX=2·1″, calculate the length of AY.

(ii) If AB=2·0″, AC=1·5″, and AY=0·9″, calculate the length of BX.

(iii) If X divides AB in the ratio 8 : 3, and if AC=8·8 cm., find AY, YC.

3. ABC is a triangle, and XY is drawn parallel to BC, cutting the other sides produced at X and Y.

(i) If AB=4·5 cm., AC=3·5 cm., and AX=7·2 cm., find by calculation and measurement the length of AY.

(ii) If X divides AB externally in the ratio 11 : 4, and if AC=4·9 cm., find the segments of AC.

(Theoretical.)

4. *Three parallel straight lines cut any two transversals proportionally.*

5. The straight line which joins the middle points of the oblique sides of a trapezium is parallel to the parallel sides.

6. Two triangles ABC, DBC stand on the same side of the common base BC; and from any point E in BC lines are drawn parallel to BA, BD, meeting AC, DC in F and G. Shew that FG is parallel to AD.

7. In a triangle ABC a transversal is drawn to cut the sides BC, CA, AB (produced if necessary) at D, E, and F respectively, and it makes equal angles with AB and AC; prove that

$$BD : CD = BF : CE.$$

EXERCISES ON THEOREM 61.

(Numerical and Graphical.)

1. Draw a triangle ABC, making $a=1\cdot5''$, $b=2\cdot4''$, and $c=3\cdot6''$. Bisect the angle A, internally and externally, by lines which meet BC and BC produced at X and Y.

Measure BX, XC; BY, YC; hence evaluate and compare the ratios

$$\frac{BX}{XC},\ \frac{BY}{YC},\ \frac{BA}{AC}.$$

2. In the triangle ABC, $a=3\cdot5$ cm., $b=5\cdot4$ cm., $c=7\cdot2$ cm.; and the internal and external bisectors of the $\angle$A meet BC at X and Y.

Calculate the lengths of the segments into which the base is divided at X and Y respectively; and verify your results graphically.

3. Frame constructions, based upon Theorem 61,

(i) to trisect a straight line of given length;

(ii) to divide a given line internally and externally in the ratio 3 : 2.

(Theoretical.)

4. AD is a median of the triangle ABC; and the angles ADB, ADC are bisected by lines which meet AB, AC at E and F respectively. Shew that EF is parallel to BC.

5. ABCD is a quadrilateral: shew that if the bisectors of the angles A and C meet in the diagonal BD, the bisectors of the angles B and D will meet on AC.

6. Employ Theorem 61 to shew that in any triangle

(i) the internal bisectors of the three angles are concurrent;

(ii) the external bisectors of two angles and the internal bisector of the third angle are concurrent.

7. If I is the in-centre of the triangle ABC, and if AI is produced to meet BC at X, shew that

$$AI : IX = AB + AC : BC.$$

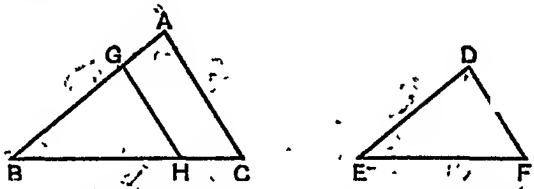
8. *Given the base of a triangle and the ratio of the other sides, find the locus of the vertex.*

9. Construct a triangle, having given the base, the ratio of the other sides, and the vertical angle.

EQUIANGULAR TRIANGLES.

THEOREM 62. [Euclid VI. 4.]

If two triangles are equiangular to one another, their corresponding sides are proportional.



Let the $\triangle^s$ ABC, DEF have the $\angle^s$ A and B respectively equal to the $\angle^s$ D and E; and consequently the $\angle$ C equal to the $\angle$ F.

It is required to prove that

AB : DE = BC : EF = CA : FD.

Proof. Apply the $\triangle$ DEF to the $\triangle$ ABC, so that E falls on B, and EF along BC;

then since the $\angle$ E = the $\angle$ B, ED will fall along BA.

Let G and H be the points at which D and F fall respectively; so that GBH represents the $\triangle$ DEF in its new position.

Now, by hypothesis, the $\angle$ D = the $\angle$ A;

that is, the ext. $\angle$ BGH = the int. opp. $\angle$ BAC;

$\therefore$ GH is par^l to AC.

Hence BA : BG = BC : BH; *Theor.* 60, *Cor.*

that is, AB : DE = BC : EF.

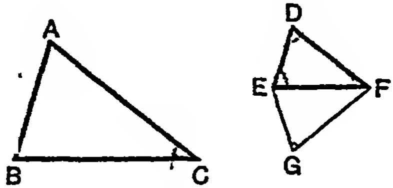
Similarly, by applying the $\triangle$ DEF to the $\triangle$ ABC, so that F falls on C, and FE, FD along CB, CA, it may be shewn that

BC : EF = CA : FD.

Hence finally, AB : DE = BC : EF = CA : FD. Q.E.D.

THEOREM 63. [Euclid VI. 5.]

If two triangles have their sides proportional when taken in order, the triangles are equiangular to one another, and those angles are equal which are opposite to corresponding sides.



In the $\triangle^s$ ABC, DEF, let

$$AB : DE = BC : EF = CA : FD.$$

It is required to prove that the $\triangle^s$ ABC, DEF *are equiangular to one another.*

At E in FE make the $\angle$ FEG equal to the $\angle$ B;
and at F in EF make the $\angle$ EFG equal to the $\angle$ C.
$\therefore$ the remaining $\angle$ EGF = the remaining $\angle$ A.

Proof. Since the $\triangle^s$ ABC, GEF are equiangular to one another,

$\therefore$ AB : GE = BC : EF. *Theor.* 62.

But, by hypothesis, AB : DE = BC : EF;
$\therefore$ AB : GE = AB : DE.
$\therefore$ GE = DE.
Similarly GF = DF.
Then in the $\triangle^s$ GEF, DEF,

because $\begin{cases} GE = DE, \\ GF = DF, \\ \text{and } EF \text{ is common}; \end{cases}$

the triangles are identically equal; *Theor.* 7
$\therefore$ the $\angle$ DEF = the $\angle$ GEF
= the $\angle$ B;
and the $\angle$ DFE = the $\angle$ GFE
= the $\angle$ C.
$\therefore$ the remaining $\angle$ D = the remaining $\angle$ A;
that is, the $\triangle$ DEF is equiangular to the $\triangle$ ABC. Q.E.D.

EXERCISES ON EQUIANGULAR TRIANGLES.

(*Numerical and Graphical. The results are to be obtained by calculation and checked graphically.*)

1. In a triangle ABC, XY is drawn parallel to BC, cutting the other sides at X and Y:

(i) If AB=2·5″, AC=2·0″, AX=1·5″; find AY.

(ii) If AB=3·5″, AC=2·1″, AY=1·2″; find AX.

(iii) If AB=4·2 cm., AX=3·6 cm., AY=6·6 cm.; find AC.

2. In the figure of the last example:

(i) If AB=2·4″, BC=3·6″, AX=1·4″; find XY.

(ii) If BC=7·7 cm., XY=5·5 cm., AX=4·5 cm.; find AB.

3. In the triangle ABC, a=3·0″, b=3·6″, c=4·2″; and QR, drawn parallel to AC, measures 3·0″. Find the remaining sides of the triangle QBR.

4. ABC is a triangle in which a=8 cm., b=7 cm., and c=10 cm. In AB a point P is taken 4 cm. from A, and PQ is drawn parallel to BC. Find the lengths of PQ and QC.

5. The sides of a triangular field are 400 yards, 350 yards, and 300 yards respectively. In a plan of the field the greatest side measures 2·4″; find the lengths of the other sides.

6. XY is drawn parallel to BC, the base of the triangle ABC. If AX=8⅓ ft., XY=3⅓ ft., AY=6 ft. 2 in., and XB=4¼ ft.; calculate the sides of the triangle ABC.

7. The triangle ABC is right-angled at C; and from P, a point in the hypotenuse, PQ is drawn parallel to AC.

If AC=1¼″, BC=3″, and PQ=½″; find BQ, BP, and AP.

8. In a triangle ABC, AD is the perpendicular from A on BC; and through X, a point in AD, a parallel is drawn to BC, meeting the other sides in P, Q.

If BC=9 cm., AD=8 cm., DX=3 cm.; find PQ.

9. In the triangle ABC, a=2·0 cm., b=3·5 cm., c=4·5 cm. BD and CE are drawn from the ends of the base to the opposite sides, and they intersect in P.

If EP : PC=DP : PB=2 : 5,

find the lengths of ED, AD, and DC.

EXERCISES ON EQUIANGULAR TRIANGLES.

(*Theoretical.*)

1. Shew that the straight line which joins the middle points of two sides of a triangle is

(i) parallel to the third side; (ii) one-half the third side.

2. In the trapezium ABCD, AB is parallel to DC, and the diagonals intersect at O: shew that

OA : OC = OB : OD.

If AB = 2DC, shew that O is a point of trisection on both diagonals.

3. If three concurrent straight lines are cut by two parallel transversals in A, B, C, and P, Q, R respectively; prove that

AB : BC = PQ : QR.

4. ABCD is a parallelogram, and from D a straight line is drawn to cut AB at E, and CB produced at F. In this figure name three triangles which are equiangular to one another; and shew that

DA : AE = FB : BE = FC : CD.

5. In the side AC of a triangle ABC any point D is taken: shew that if AD, DC, AB, BC are bisected in E, F, G, H respectively, then EG is equal to HF.

6. AB and CD are two parallel straight lines; E is the middle point of CD; AC and BE meet at F, and AE and BD meet at G: shew that FG is parallel to AB.

7. AB is a diameter of a circle, and through A any straight line is drawn to cut the circumference in C and the tangent at B in D: shew that

(i) the Δ^s CAB, BAD are equiangular to one another;

(ii) AC, AB, AD are three proportionals;

(iii) the rect. AC, AD is constant for all positions of AD.

8. If through any point X within a circle two chords AB, CD are drawn, and AC, BD joined; shew that

(i) the Δ^s AXC, DXB are equiangular to one another;

(ii) AX : DX = XC : XB.

Hence obtain an alternative proof of Theorem 57.

9. If from an external point X a tangent XT and a secant XAB are drawn to a circle, and AT, TB joined; shew that

(i) the Δ^s AXT, TXB are equiangular to one another;

(ii) XA : XT = XT : XB.

Hence obtain an alternative proof of Theorem 58.

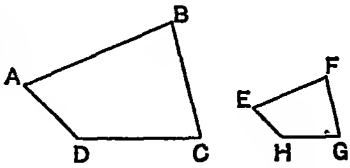
DEFINITIONS.

1. Two rectilineal figures are said to be **equiangular** to one another when the angles of the first, taken in order, are equal respectively to those of the second, taken in order.

2. Rectilineal figures are said to be **similar** when they are equiangular to one another, and also have their corresponding sides proportional.

Thus the two quadrilaterals ABCD, EFGH are similar if the angles at A, B, C, D are respectively equal to those at E, F, G, H, and if the following proportions hold:

$$\frac{AB}{EF}=\frac{BC}{FG}=\frac{CD}{GH}=\frac{DA}{HE}.$$



3. Similar figures are said to be **similarly described** with regard to two sides, when these sides correspond.

NOTE ON SIMILAR FIGURES.

Similar figures may be described as those which have the *same shape*. For this, *two* conditions are necessary:

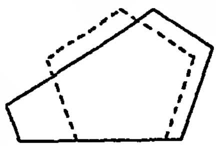
(i) *the figures must have their angles equal each to each, taken in order;*

(ii) *their corresponding sides must be proportional.*

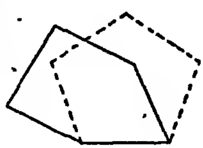
In the case of *triangles* we have learned that these conditions are not independent, for each follows from the other: thus

(i) if the triangles are *equiangular to one another*, Theorem 62 proves *that their corresponding sides are proportional;*

(ii) if the triangles have their *sides proportional*, Theorem 63 proves *that they are equiangular to one another.*

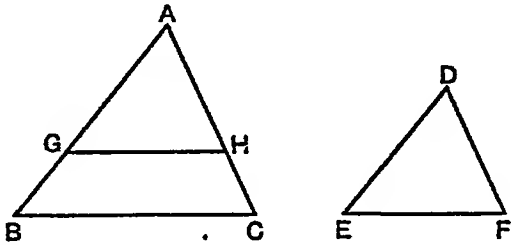


This, however, is not necessarily the case with rectilineal figures of more than three sides. For example, the first diagram in the margin shews two figures which are equiangular to one another, but which clearly have not their sides proportional; while the figures in the second diagram have their sides proportional, but are not equiangular to one another.



Theorem 64. [Euclid VI. 6.]

If two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportionals, the triangles are similar.



In the $\triangle^s$ ABC, DEF, let the $\angle$ A = the $\angle$ D,

and let AB : DE = AC : DF.

It is required to prove that the $\triangle^s$ ABC, DEF *are similar.*

Proof. Apply the $\triangle$ DEF to the $\triangle$ ABC, so that D falls on A, and DE along AB;

then because the $\angle$ EDF = the $\angle$ BAC, DF must fall along AC.

Let G and H be the points at which E and F fall respectively; so that AGH represents the $\triangle$ DEF in its new position.

Now, by hypothesis, AB : DE = AC : DF;

that is, AB : AG = AC : AH;

hence GH is par^l to BC. *Theor.* 60, *Cor*

$\therefore$ the ext. $\angle$ AGH, namely the $\angle$ E, = the int. opp. $\angle$ ABC;

and the ext. $\angle$ AHG, namely the $\angle$ F, = the int. opp. $\angle$ ACB.

Hence the $\triangle^s$ ABC, DEF are equiangular to one another, so that their corresponding sides are proportional; *Theor.* 62

that is, the $\triangle^s$ ABC, DEF are similar.

Q.E.D.

*Theorem 65. [Euclid VI. 7.]

If two triangles have one angle of the one equal to one angle of the other, and the sides about another angle in one proportional to the corresponding sides of the other, then the third angles are either equal or supplementary; and in the former case the triangles are similar.

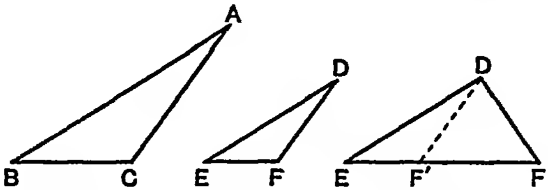


Fig. 1. Fig. 2. Fig. 3.

In the $\triangle^s$ ABC, DEF, let the $\angle$B = the $\angle$E; and let the sides about the $\angle^s$ A and D be proportional,

namely AB : DE = AC : DF.

It is required to prove that

either the $\angle$C = the $\angle$F, [as in Figs. 1 and 2];

or the $\angle$C = *the supplement of the* $\angle$F. [Figs. 1 and 3.]

Proof. (i) If the $\angle$A = the $\angle$D, [Figs. 1 and 2],

then the $\angle$C = the $\angle$F; *Theor.* 16.

and the $\triangle^s$ are equiangular, and therefore similar.

(ii) But if the $\angle$A is not equal to the $\angle$EDF [Figs. 1 and 3]

let the $\angle$EDF′ = the $\angle$A.

Then the $\triangle^s$ ABC, DEF′ are equiangular to one another;

∴ AB : DE = AC : DF′.

But by hypothesis, AB : DE = AC : DF;

∴ AC : DF′ = AC : DF.

∴ DF′ = DF.

∴ the $\angle$DFF′ = the $\angle$DF′F.

But the $\angle$C = the $\angle$DF′E *Proved.*

= the supplement of the $\angle$DF′F

= the supplement of the $\angle$DFE.

Q.E.D.

EXERCISES ON SIMILAR TRIANGLES.

(*Theoretical.*)

1. In a triangle ABC, prove that any straight line parallel to the base BC and intercepted by the other two sides is bisected by the median drawn from the vertex A.

2. Two triangles ABC, A′B′C′ are equiangular to one another;
if p, p' denote the perpendiculars from A, A′ to the opp. sides,
R, R′...............circum-radii;
r, r' ....in-radii;

prove that each of the ratios $\frac{p}{p'}$, $\frac{R}{R'}$, $\frac{r}{r'}$ is equal to the ratio of any pair of corresponding sides.

3. Prove that the radius of the circle which passes through the mid-points of the sides of a triangle is half the circum-radius.

4. If two straight lines AB, CD intersect at X, so that

$$XA : XC = XD : XB;$$

(i) shew by Theorem 64 that the $\triangle^s$ AXD, CXB are similar:

(ii) hence prove the points A, D, B, C concyclic.

5. A, B, C are three collinear points, and from B and C two parallel lines BP, CQ are drawn in the same sense, so that

$$PB : QC = AB : AC;$$

shew by Theorem 64 that the points A, P, Q are collinear.

6. If in two triangles ABC, A′B′C′, the $\angle B =$ the $\angle B'$, and $\frac{c}{c'} = \frac{b}{b'}$; what conclusion may be drawn?

Shew by diagrams how this conclusion is affected, if it is also given that
(i) c is less than b,
(ii) c is equal to b,
(iii) c is greater than b.

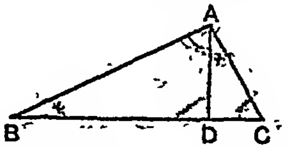
7. ABCD is a parallelogram; P and Q are points in a straight line parallel to AB; PA and QB meet at R, and PD and QC meet at S: shew that RS is parallel to AD.

8. In a triangle ABC the bisector of the vertical angle A meets the base at D and the circumference of the circum-circle at E; if EC is joined, shew that the triangles BAD, EAC are similar; and hence prove that

$$AB \cdot AC = AE \cdot AD.$$

THEOREM 66. [Euclid VI. 8.]

In a right-angled triangle, if a perpendicular is drawn from the right angle to the hypotenuse, the triangles on each side of it are similar to the whole triangle and to one another.



Let BAC be a triangle right-angled at A, and let AD be drawn perp. to BC.

It is required to prove that the △ˢ BDA, ADC *are similar to the* △ BAC *and to one another.*

In the △ˢ BDA, BAC,
the ∠ BDA = the ∠ BAC, being rt. angles,
and the ∠ B is common to both;
∴ the remaining ∠ BAD = the remaining ∠ BCA; *Theor.* 16
hence the △ BDA is equiangular to the △ BAC;
∴ their corresponding sides are proportional;
∴ the △ˢ BDA, BAC are similar.

In the same way it may be proved that the △ˢ ADC, BAC are similar.

Hence the △ˢ BDA, ADC, having their angles severally equal to those of the △ BAC, are equiangular to one another;
∴ they are similar.

Q.E.D.

COROLLARY. (i) Because the △ˢ DBA, DAC are similar,
∴ DB : DA = DA : DC:
that is, DA is a mean proportional between DB and DC;
and $DA^2 = DB . DC$.

(ii) Because the △ˢ BCA, BAD are similar,
∴ BC : BA = BA : BD;
hence $BA^2 = BC . BD$.

(iii) Because the △ˢ CBA, CAD are similar,
∴ CB : CA = CA : CD;
hence $CA^2 = CB . CD$.

EXERCISES.

(*Miscellaneous Examples on Theorems* 62–66.)

1. ABC is an equilateral triangle of which each side $=a$. In BC, produced both ways, two points P and Q are taken, such that $BP=CQ=a$, and AP, AQ are joined. Shew that

(i) $PQ : PA = PA : PB$.

(ii) $PA^2 = 3a^2$.

2. ABC is a triangle right-angled at A, and AD is drawn perpendicular to BC: if AB, AC measure respectively 4″ and 3″, shew that the segments of the hypotenuse are 3·2″ and 1·8″.

3. ABC is a triangle right-angled at A, and a perpendicular AD is drawn to the hypotenuse BC; shew (i) by Theorem 25, (ii) by Theorem 66, that

$$BC \cdot AD = AB \cdot AC.$$

4. ABC is a triangle right-angled at A, and AC′ is drawn perpendicular to the hypotenuse, also C′A′ is drawn parallel to CA. If AC=15 cm., and AB=20 cm., shew that AC′=12 cm., and C′A′=9·6 cm.

5. At the extremities of a diameter of a circle, whose centre is C and radius r, tangents are drawn: these are cut in Q and R by any third tangent whose point of contact is P. Shew that

(i) QR subtends a right angle at C;

(ii) $PQ \cdot PR = r^2$.

6. Two circles of radii r and r' respectively have external contact at A, and a common tangent touches them at P and Q. Shew that

(i) PQ subtends a right angle at A; [Ex. 9. p. 187]

(ii) $PQ^2 = 4rr'$.

[Produce PA, QA to meet the circumferences at X and Y, and prove the triangles PAY, XAQ right-angled and similar.]

7. Two circles touch one another externally at A, and a common tangent PQ is produced to meet the line of centres at S. Shew that, if PA, AQ are joined,

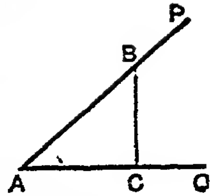
(i) the triangles SAP, SQA are similar;

(ii) $SA^2 = SP \cdot SQ$.

8. Two circles intersect at A and B; and at A tangents are drawn, one to each circle, to meet the circumferences at C and D: shew that if BC, BD are joined,

$$BC : BA = BA : BD.$$

THE TRIGONOMETRICAL RATIOS.



1. Let PAQ be any acute angle; in AP, one of the arms of the angle, take a point B, and draw BC perp. to AQ.

Then with reference to the $\angle A$ in the right-angled $\triangle$ BAC, the following definitions are used.

The ratio $\frac{BC}{AB}$, or $\frac{\text{opposite side}}{\text{hypotenuse}}$, is called the sine of the $\angle A$.

The ratio $\frac{AC}{AB}$, or $\frac{\text{adjacent side}}{\text{hypotenuse}}$, ... cosine of the $\angle A$.

The ratio $\frac{BC}{AC}$, or $\frac{\text{opposite side}}{\text{adjacent side}}$, ... tangent of the $\angle A$.

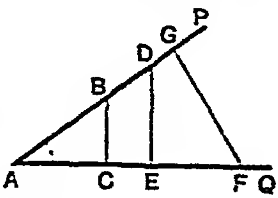
The reciprocals of these ratios are known respectively as the cosecant the secant, and the cotangent of A.

These six ratios are called the trigonometrical ratios of the $\angle A$, and are usually expressed in the following shorter form.

$$\sin A = \frac{BC}{AB}, \quad \cos A = \frac{AC}{AB}, \quad \tan A = \frac{BC}{AC},$$

$$\operatorname{cosec} A = \frac{AB}{BC}, \quad \sec A = \frac{AB}{AC}, \quad \cot A = \frac{AC}{BC}.$$

NOTE. The squares of these ratios, namely $(\sin A)^2$, $(\cos A)^2$, ... are usually written in the form $\sin^2 A$, $\cos^2 A$, ...



2. In the adjoining figure, let BC, DE be perps. to AQ from points in AP, and let FG be perp. to AP from a point F in AQ.

Then the $\triangle$s BAC, DAE, FAG are similar, so that

$$\frac{BC}{AB} = \frac{DE}{AD} = \frac{FG}{AF}.$$

But these ratios express the value of $\sin A$ according as it is determined from the $\triangle$ BAC, the $\triangle$ DAE, or the $\triangle$ FAG.

Thus $\sin A$ is unaltered so long as the $\angle A$ remains the same. A similar proof holds for each of the trigonometrical ratios, shewing that they depend only on the size of the angle and not upon the lengths of its arms

EXERCISES.

1. In a triangle ABC, right-angled at C, $a=8$, $b=15$; find c, and write down the values of sin A, cos A, and tan A.

2. In a right-angled triangle, the sides containing the right angle are 35 and 12: find the hypotenuse, and write down all the trigonometrical ratios of the smallest angle.

3. If A is any acute angle, shew that Theorem 29 may be made to assume either of the forms:

$$\text{(i)}\ \sin^2 A+\cos^2 A=1;\quad \text{(ii)}\ \sec^2 A=1+\tan^2 A.$$

4. ABCD is a quadrilateral in which the diagonal AC is at right angles to each of the sides AB, CD. If AB=1·5 cm., AC=3·6 cm., AD=8·5 cm., draw the figure, and find sin ABC, tan ACB, cos CDA, tan DAC.

5. If A is any acute angle, shew that

$$\text{(i)}\ \sin(90°-A)=\cos A;\quad \text{(ii)}\ \tan(90°-A)=\cot A.$$

6. Construct an acute angle whose sine is 0·6. [See Prob. 10, p. 83.] Measure the angle with your protractor and give its value to the nearest degree.

7. Construct an acute angle A from each of the following data:

(i) tan A=0·7; (ii) cos A=0·9; (iii) sin A=0·71.

In each case measure the angle to the nearest degree.

8. Construct an acute angle A such that tan A=1·6. Measure the angle A, and ascertain by measurement and by calculation the value of cos A.

9. Prove the following results

$$\text{(i)}\ \sin 45°=\cos 45°=\frac{1}{\sqrt{2}};\quad \text{(ii)}\ \sin 60°=\cos 30°=\frac{\sqrt{3}}{2}.$$

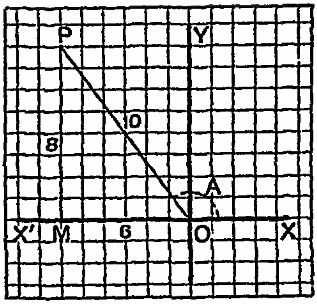
[See Ex. 11, p. 123, and Ex. 14, p. 124.]

10. Construct a triangle ABC, right-angled at C, having the hypotenuse 10 cm. in length, and tan A=0·81. Measure AC and the angle A; and find the values of sin A and cos A.

11. Draw a right-angled triangle ABC from the following data:

$$\tan A=0{\cdot}7,\quad \angle C=90°,\quad b=2{\cdot}8 \text{ cm.}$$

Measure c and the $\angle$A.



3. The definitions on page 270 may be extended to obtuse angles as follows:

Let XOX′ be a straight line, and let OY be perp. to it.

Let the angle A be traced by the revolution about O of the line OP which starts from the position OX.

Draw PM perp. to X′OX, thus forming a right-angled triangle POM. Then *whatever the position of* OP, the trigonometrical ratios of the angle A through which OP has turned are thus defined:

$$\sin A=\frac{PM}{OP},\quad \cos A=\frac{OM}{OP},\quad \tan A=\frac{PM}{OM},$$

with the understanding that OM *is to be considered positive when it is to the right of* OY, *and negative when to the left of* OY. [Compare p. 133.]

For example, in the above figure,

$$\sin A=\frac{PM}{OP}=\frac{8}{10}=\cdot 8.$$

$$\cos A=\frac{OM}{OP}=\frac{-6}{10}=-\cdot 6.$$

$$\tan A=\frac{PM}{OM}=\frac{8}{-6}=-\frac{4}{3}.$$

EXAMPLE. *To express trigonometrically*

(i) *the perpendicular from the vertex of a triangle on the base:*

(ii) *the projection of one side on another.*

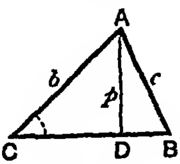


Fig. 1.

(i) In both Figs., $\frac{AD}{AC}=\sin C,$

the sine of C being positive in each case.

$$\therefore\ p=b\sin C.$$

(ii) In Fig. 1, $\frac{CD}{CA}=\cos C.$

In Fig. 2, $\frac{CD}{CA}$ also represents $\cos C$,

if CD *is considered negative.*

∴ *numerically*

$CD=+b\cos C$ in Fig. 1.

$CD=-b\cos C$ in Fig. 2.

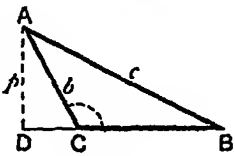


Fig. 2.

SOME GEOMETRICAL RESULTS EXPRESSED IN TRIGONOMETRICAL FORM.

[*The diagrams referred to are those of the preceding example:*]

1. In both Figs., $p = b \sin C$.

Similarly it may be proved that $p = c \sin B$.

Hence $b \sin C = c \sin B$; $\therefore \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

Similarly $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C};$$

that is, *the sides of a triangle are proportional to the sines of the opposite angles.*

2. From this property of a triangle deduce Theorem 62

3. In both Figs.

$$\text{area of } \Delta\, ABC = \tfrac{1}{2}BC\,.\,AD = \tfrac{1}{2}ap;$$

and $$p = b \sin C; \quad \therefore \Delta = \tfrac{1}{2}ab \sin C$$

Similarly $$\Delta = \tfrac{1}{2}bc \sin A = \tfrac{1}{2}ca \sin B = \tfrac{1}{2}ab \sin C.$$

4. Express in trigonometrical form the area of

(i) a parallelogram, given two adjacent sides and the included angle

(ii) a rhombus, given one side and one angle.

5. Shew that the circum-radius of a triangle is given by the formula

$$R = \frac{a}{2 \sin A} = \frac{abc}{4\Delta}.$$

6. In Fig. 1, we have

$$AB^2 = BC^2 + CA^2 - 2BC\,.\,CD. \qquad \textit{Theor. 55.}$$

In Fig. 2, we have

$$AB^2 = BC^2 + CA^2 + 2BC\,.\,CD. \qquad \textit{Theor. 54.}$$

Now in Fig. 1, $CD = +b \cos C$;

and in Fig. 2, $CD = -b \cos C$.

Hence in both cases we have, on substitution,

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

Similarly it may be shewn that

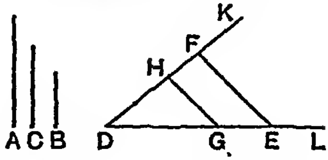
$$a^2 = b^2 + c^2 - 2bc \cos A.$$
$$b^2 = c^2 + a^2 - 2ca \cos B.$$

PROBLEMS.

PROBLEM 35.

To find the **fourth proportional** *to three given straight lines.*

Let A, B, C be the three given st. lines, to which the fourth proportional is required.



Construction. Draw two st. lines DL, DK of indefinite length, containing any angle.

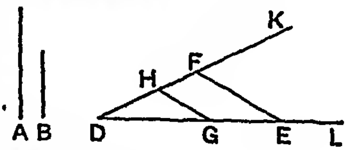
From DL cut off DG equal to A, and GE equal to B;
and from DK cut off DH equal to C.
Join GH.
Through E draw EF par[1] to GH.
Then HF is the fourth proportional to A, B, C.

Proof. Because GH is par[1] to EF, a side of the △ DEF;
∴ DG : GE = DH : HF.
But DG = A, GE = B, and DH = C;
∴ A : B = C : HF;
that is, HF is the fourth proportional to A, B, C.

PROBLEM 36.

To find the **third proportional** *to two given straight lines.*

Let A, B be the two lines to which the third proportional is required.

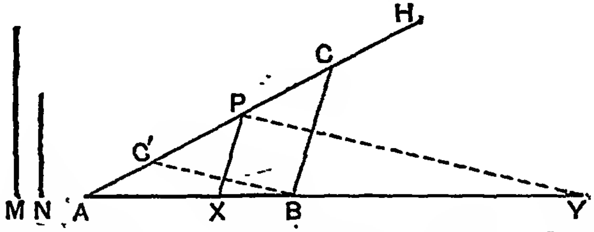


Construction. Draw two st. lines DL, DK.
From DL cut off DG equal to A, and GE equal to B;
and from DK cut off DH also equal to B.
Join GH.
Through E draw EF par[1] to GH.
Then HF is the third proportional to A, B.

Proof. As above, in Problem 35.

PROBLEM 37.

To divide a given straight line **internally** *and* **externally** *in a* **given ratio.**



Let AB be the st. line to be divided internally and externally in the ratio M : N.

Construction. From A draw a st. line AH at any angle with AB.

From AH cut off AP equal to M.

From PH and PA cut off PC and PC′, each equal to N.

Join BC, BC′.

Through P draw PX par[1] to BC, and PY par[1] to BC′.

Then AB is divided internally at X, and externally at Y in the ratio M : N.

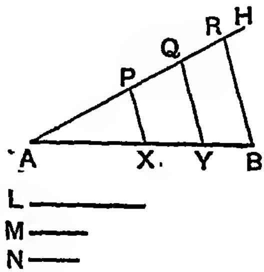
Proof. (i) Because PX is par[1] to BC, a side of the $\triangle$ ABC,

$$\therefore \; AX : XB = AP : PC$$
$$= M : N.$$

(ii) Because PY is par[1] to BC′, a side of the $\triangle$ ABC′,

$$\therefore \; AY : YB = AP : PC'$$
$$= M : N.$$

COROLLARY. By a similar process a st. line AB may be divided internally into segments proportional to three lines L, M, N.

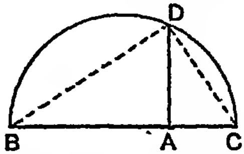


Construction. Draw AH, and from it cut off AP, PQ, QR equal respectively to L, M, N. Join RB; and through P and Q draw PX, QY par[l] to BR.

Then evidently $\quad AX : L = XY : M = YB : N.$

Problem 38.

To find the mean proportional *between two given straight lines.*



Let AB, AC be the two given st. lines between which the mean proportional is to be found.

Construction. Place AB, AC in a straight line, and in opposite senses; and on BC describe the semi-circle BDC.

From A draw AD at rt. angles to BC, to cut the O^{ce} at D.

Then AD is the mean proportional between AB and AC.

Proof. Join BD, DC.

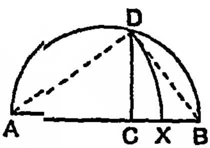
Now the ∠ BDC, being in a semi-circle, is a rt. angle.

And because in the right-angled △ BDC, DA is drawn from the rt. angle perp. to the hypotenuse,

∴ the △^{s} ABD, ADC are similar; *Theor.* 66.

∴ AB : AD = AD : AC;

that is, AD is the mean proportional between AB and AC.



NOTE. If the given lines AB, AC are placed in he same sense, the mean proportional between hem may be cut off from AB by the following seful construction.

On AB draw a semi-circle; and from C draw CD perp. to AB to cut the O^{ce} at D. From AB cut off AX equal to AD.

Then AX is the mean proportional between AB and AC.

The △^{s} ABD, ADC are similar, *Theor.* 66.

∴ AB : AD = AD : AC;

that is, AB : AX = AX : AC.

GRAPHICAL EVALUATION OF A QUADRATIC SURD.

EXAMPLE. Find the approximate value of (i) $\sqrt{5}$, (ii) $\sqrt{21}$.

(i) $\sqrt{5}=\sqrt{5\times 1}$. Hence take AB, AC respectively to represent 5 and 1 in terms of any convenient unit, and find AD the mean proportional between them.

Then since

$$AB : AD = AD : AC,$$
$$\therefore\ AD^2 = AB \,.\, AC$$
$$= 5 \times 1 = 5.$$
$$\therefore\ AD = \sqrt{5}.$$

Thus *by measuring* AD, the value of $\sqrt{5}$ is roughly found to be 2·24.

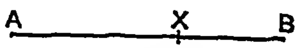
(ii) $\sqrt{21}=\sqrt{7\times 3}$. Here take AB, AC equal to 7 cm. and 3 cm. respectively, and proceed as before.

NOTE. Factors should be chosen so as to give convenient lengths for AB, AC.

$$e.g.\ \sqrt{23}=\sqrt{2{\cdot}3\times 10};\quad \sqrt{11}=\sqrt{2{\cdot}2\times 5}.$$

DEFINITION.

A straight line is said to be divided in extreme and mean ratio, when the whole is to the greater segment as the greater segment is to the less.



Thus AB is divided at X in extreme and mean ratio,

when $$AB : AX = AX : XB;$$

from which it follows that

$$AB \,.\, BX = AX^2;$$

or, *the rectangle contained by the whole line and one part is equal to the square on the other part.*

Hence a straight line may be divided in extreme and mean ratio by Problem 33. For Construction and Proof see page 240.

EXERCISES.

1. Find graphically, testing your results by arithmetic:

 (i) The 4th proportional to 2·4″, 1·5″, 1·6″.

 (ii) The 3rd proportional to 2·5″ and 1·5″.

 (iii) The mean proportional between 7·2 cm. and 5·0 cm.

2. Divide a line, 2·0″ in length, internally and externally in the ratio 7 : 3; and in each case find the segments by measurement and calculation.

3. Obtain graphically the unknown term in the following statements of proportion; and check your result by arithmetic:

 (i) $1{\cdot}25 : x = 1{\cdot}0 : 1{\cdot}6$. [Take 1″ as the unit of length.]

 (ii) $x : 4{\cdot}2 = 4{\cdot}2 : 6{\cdot}3$. [Take 1 cm. as the unit of length.]

 (iii) $x : 16 = 25 : x$. [Let 1″ represent 10.]

4. Divide a line, 7·2 cm. in length, into three parts proportional to the numbers 2, 3, 4. Test your construction by measurement and calculation.

5. Divide a line, 3·9″ in length, into three parts, so that the second $=\frac{2}{3}$ of the first, and the third $=\frac{3}{4}$ of the second.

6. On a side of 1·5″ draw a rectangle equal in area to a square on a side of 2″. Measure the other side of the rectangle.

7. Find graphically the approximate values of

 (i) $\sqrt{3}$; (ii) $\sqrt{10}$; (iii) $\sqrt{\frac{14}{5}}$.

8. Determine by geometrical constructions the approximate values of the following expressions, in each case verifying your drawing arithmetically:

 (i) $\dfrac{3{\cdot}5\times 2{\cdot}4}{2{\cdot}8}$; (ii) $\dfrac{6{\cdot}84}{2{\cdot}13}$; (iii) $\dfrac{2{\cdot}71\times 1{\cdot}26}{1{\cdot}51}$.

9. Draw a triangle ABC from each of the following sets of data, and in each case calculate and measure the lengths of the sides:

 (i) The perimeter $=4{\cdot}8''$; and $\dfrac{a}{3}=\dfrac{b}{4}=\dfrac{c}{5}$.

 (ii) The perimeter $=11{\cdot}1$ cm.; and $a=\frac{5}{6}b$, $b=\frac{4}{5}c$.

 (iii) The perimeter $=11{\cdot}8$ cm.; and $\dfrac{A}{1}=\dfrac{B}{2}=\dfrac{C}{3}$.

 (iv) $a=4{\cdot}0''$, $A=90^\circ$; and $b : c = 5 : 3$.

EXERCISES.

(Proportion applied to the calculation of Heights and Distances.)

1. A field is represented in a plan by a triangle ABC, in which $a=8$ cm., $b=5{\cdot}6$ cm., $c=6{\cdot}4$ cm. If the greatest side of the field is 200 metres, find the lengths of the other sides.

A fence, run across the field, is represented in the plan by a line PQ parallel to BC drawn from a point P in AB distant 4·0 cm. from A. Find the length of the fence.

2. A's speed is to B's in the ratio 8 : 7; find *graphically* by how much A would beat B in a 100 yards' race, supposing each man to run throughout at a uniform rate.

3. On a map in which 1″ represents 25 miles, three places A, B, and C are marked. Of these, B appears N.W. of A at a distance 0·8″; and C appears N.E. of A at a distance 1·5″. Find the actual distance between B and C.

4. A man, whose height is 6 feet, standing 32 feet from a lamp-post, observes that his shadow cast by the light is 8 feet in length: how high is the light above the ground, and how long would be the shadow of a boy 5 feet in height standing 20 feet from the post?

5. A man 6 feet in height, standing 15 feet from a lamp-post, observes that his shadow cast by the light is 5 feet in length: how high is the light, and how long would his shadow be if he were to approach 8 feet nearer to the post?

6. To find the width of a canal a rod is fixed vertically on the bank so as to shew $4\frac{1}{4}$ feet of its length. The observer, whose eye is 5 ft. 8 in. above the ground retires at right angles from the canal until he sees the top of the rod in a line with the further bank. If his distance from the nearer bank is now 20 feet, what is the width of the canal?

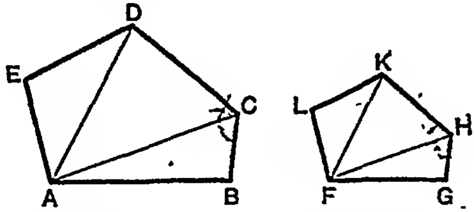
7. A man, wishing to ascertain the height of a tower, fixes a staff vertically in the ground at a distance of 27 ft. from the tower. Then, retiring 3 ft. farther from the tower, he sees the top of the staff in line with the top of the tower. If the observer's eye and the top of the staff are respectively 5 ft. 4 in. and 12 ft. above the ground, find the height of the tower.

8. A person due S. of a lighthouse observes that his shadow cast by the light at the top is 24 feet long. On walking 100 yards due E. he finds his shadow to be 30 feet long. Supposing him to be 6 feet high, find the height of the light from the ground.

SIMILAR FIGURES.

√THEOREM 67.

Similar polygons can be divided into the same number of similar triangles; and the lines joining corresponding vertices in each figure are proportional.



Let ABCDE, FGHKL be similar polygons, the vertex A corresponding to the vertex F, B to G, and so on. Let AC, AD be joined, and also FH, FK.

It is required to prove that

(i) the $\triangle^s$ ABC, FGH *are similar; as also the* $\triangle^s$ ACD, FHK, *and the* $\triangle^s$ ADE, FKL.

(ii) $AB : FG = AC : FH = AD : FK.$

Proof. (i) Since the polygons are similar,

the $\angle ABC =$ the $\angle FGH$,

and $AB : FG = BC : GH$;

$\therefore$ the $\triangle^s$ ABC, FGH are similar. *Theor.* 64.

$\therefore$ the $\angle BCA =$ the $\angle GHF$;

but because the polygons are similar,

the $\angle BCD =$ the $\angle GHK$;

$\therefore$ the $\angle ACD =$ the $\angle FHK$.

Also $AC : FH = BC : GH$, for the $\triangle^s$ ABC, FGH are similar,

$= CD : HK$, for the polygons are similar.

That is, the sides about the equal $\angle^s$ ACD, FHK are proportional,

$\therefore$ the $\triangle^s$ ACD, FHK are similar. *Theor.* 64

In the same way the $\triangle^s$ ADE, FKL are similar.

(ii) And $AB : FG = AC : FH$, from the similar $\triangle^s$ ABC, FGH;

$= AD : FK$, from the similar $\triangle^s$ CAD, HFK.

Q.E.D

NOTE. In the last Theorem the polygons have been divided into similar triangles by lines drawn from a pair of *corresponding vertices.* But this restriction is not necessary.

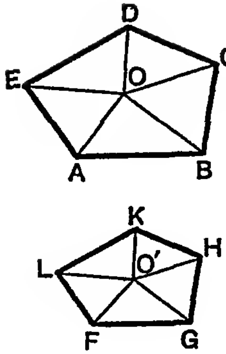
For take *any* point O in the polygon ABCDE, and join it to each of the vertices.

In the polygon FGHKL, make the ∠GFO′ equal to the ∠BAO,

and make the ∠FGO′ equal to the ∠ABO.

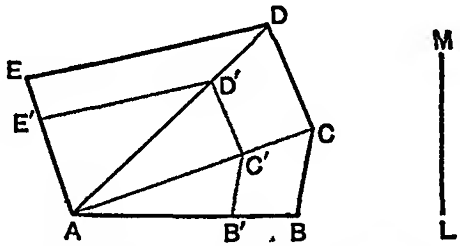
Join O′ to each vertex of the polygon FGHKL.

We leave as an exercise to the student the proof that the two polygons are thus divided into the same number of similar triangles.



PROBLEM 39. [First Method.]

On a side of given length to draw a figure similar to a given rectilineal figure.



Let ABCDE be the given figure, and LM the length of the given side; and suppose that this side is to correspond to AB.

Construction. From AB cut off AB′ equal to LM.
Join AC, AD.
From B′ draw B′C′ par[l] to BC, to cut AC at C′.
From C′ draw C′D′ par[l] to CD, to cut AD at D′.
From D′ draw D′E′ par[l] to DE, to cut EA at E′.
Then AB′C′D′E′ is the required figure.

Outline of Proof. (i) By construction the figure AB′C′D′E′ is equiangular to the figure ABCDE.

(ii) From the three pairs of similar triangles it may be shewn that

$$\frac{AB'}{AB}=\frac{B'C'}{BC}=\frac{C'D'}{CD}=\frac{D'E'}{DE}=\frac{E'A}{EA};$$

that is, corresponding sides of the polygons are proportional,

THEOREM 68.

Any two similar rectilineal figures may be so placed that the lines joining corresponding vertices are concurrent.

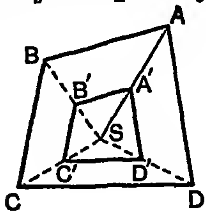


Fig. 1.

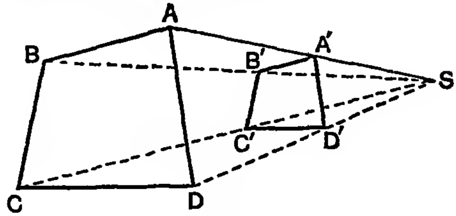


Fig. 2.

Let ABCD, A′B′C′D′ be similar figures.

Then since the $\angle$ B′ = the $\angle$ B, the figures can be so placed that A′B′, B′C′ are respectively par[l] to AB, BC. It follows, since the figures are equiangular to one another, that C′D′ is par[l] to CD, and D′A′ par[l] to DA.

It is required to prove that when corresponding sides of the given figures are parallel, then AA′, BB′, CC′, DD′ *are concurrent.*

Join AA′, and divide it externally at S in the ratio AB : A′B′.

Join SB *and* SB′: it will be shewn that SB and SB′ are in one straight line.

Proof. In the $\triangle$[s] SAB, SA′B′, since AB and A′B′ are par[l],

$\therefore$ the $\angle$ SAB = the $\angle$ SA′B′;

and, by construction, SA : SA′ = AB : A′B′;

$\therefore$ the $\triangle$[s] SAB, SA′B′ are equiangular to one another; *Theor.* 64.

$\therefore$ the $\angle$ ASB = the $\angle$ A′SB′.

Hence SB, SB′ are in the same st. line;

that is, BB′ passes through the fixed point S.

Similarly CC′ and DD′ may be shewn to pass through S.

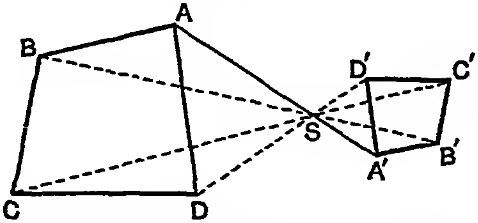
That is, AA′, BB′, CC′, DD′ are concurrent. Q.E.D.

NOTE. Observe that the joining lines AA′, BB′, CC′, DD′ are all divided externally at S in the ratio of any pair of corresponding sides of the given figures.

NOTE. In placing the given figures so that A′B′, B′C′ are respectively parallel to AB, BC, two cases arise:

(i) A′B′ and AB may have *the same sense*, as in Figs. 1 and 2;

(ii) A′B′ and AB .... *opposite senses*, as in the Fig. below.

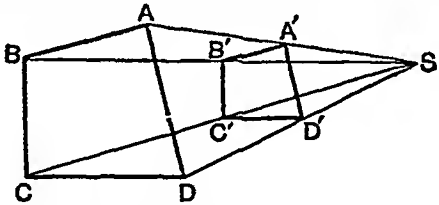


In the latter case it follows also that C′D′ is par[l] to CD, and D A′ par[l] to DA, and it may be proved as before that AA′, BB′, CC′, DD′ are concurrent; but here S divides AA′ *internally* in the ratio of corresponding sides, and the position of the figures is *transverse*.

In each case the point S is called a **centre of similarity**, or **homothetic centre**; and similar figures so placed are said to be **homothetic**.

PROBLEM 39. [Second Method.]

On a given side to draw a figure similar to a given figure.



Let ABCD be the given figure, and A′B′ the given side; and let A′B′ correspond to AB.

Construction. Place A′B′ par[l] to AB; and join AA′, BB′ by lines meeting at S.

Join SC, SD.

Through B′ draw B′C′ par[l] to BC, to meet SC at C′;
through C′ draw C′D′ par[l] to CD, to meet SD at D′.

Join A′D′.

Then A′B′C′D′ is the required figure.

The student should prove (i) that A′B′C′D′ is equiangular to ABCD, (ii) that corresponding sides of these figures are proportional. The proof is the converse of Theorem 68.

EXERCISES ON SIMILAR FIGURES.

(Numerical and Graphical.)

1. On a base AB, 6·5 cm. in length, draw a quadrilateral ABCD from the following data:

$$\angle A=80^\circ,\ \ \angle B=70^\circ,\ \ AD=4{\cdot}4\text{ cm.},\ \ BC=3{\cdot}2\text{ cm.}$$

Taking any convenient point as centre of similarity, make

(i) A reduced copy of ABCD, such that the ratio of each side to the corresponding side of ABCD is 3 : 4.

(ii) An enlarged copy of ABCD, such that the ratio of each side to the corresponding side of ABCD is 5 : 4.

2. Draw a semi-circle on a given diameter AB, and inscribe a square in it, so that two vertices may be on the arc, and the other two on AB.

If $AB=2r$, and the side of the inscribed square $=a$, shew that

$$5a^2=4r^2.$$

3. Draw a sector of a circle of radius 2·4″, the central angle being 60°; and inscribe a square in it.

If the radius of the sector $=r$, and the side of the square $=a$, calculate from measurements the ratio $a:r$.

4. In a sector of which the radius = 5 cm., and the central angle = 45°, inscribe a rectangle having its adjacent sides in the ratio 2 : 1.

Prove that two such rectangles can be drawn, and compare by measurement their greater sides.

5. Draw a triangle ABC, making $a=8$ cm., $b=7$ cm., and $c=6$ cm.

Working from the vertex A as centre of similarity, inscribe a square in the triangle, so that two of its angular points may be in the base BC, and the other two in AB, AC.

6. Draw a triangle ABC, making $a=2{\cdot}6''$, $B=110^\circ$, $C=35^\circ$.

In the triangle ABC inscribe an equilateral triangle, having

(i) one side parallel to BC;

(ii) one side parallel to any given straight line.

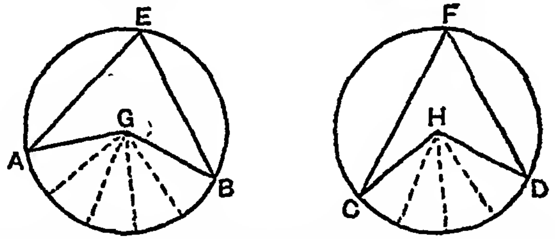
7. In a given triangle ABC inscribe a triangle similar to a given triangle DEF.

In how many ways may this be done?

8. Draw a regular hexagon ABCDEF on a side of 1·2″, and in it inscribe a square having two sides parallel to AB and DE, and its vertices on the remaining sides of the hexagon.

THEOREM 69. [Euclid VI. 33.]

In equal circles, angles, whether at the centres or circumferences have the same ratio as the arcs on which they stand.



Let ABE, CDF be equal circles; and let the $\angle^s$ AGB, CHD at the centres, and the $\angle^s$ AEB, CFD at the $\bigcirc^{ces}$, stand on the arcs AB, CD.

It is required to prove that

(i) *the* $\angle$AGB : *the* $\angle$CHD = *the arc* AB : *the arc* CD;

(ii) *the* $\angle$AEB : *the* $\angle$CFD = *the arc* AB : *the arc* CD.

Proof. Suppose the arc AB : the arc CD = $m : n$;
so that, if the arc AB is divided into m equal parts, then the arc CD may be divided into n such equal parts.

In each circle let radii be drawn to the points of division of the arcs AB, CD.

Then the $\angle^s$ AGB, CHD, in equal circles, are divided into angles which stand on equal arcs, and are therefore all equal.

And of these equal angles the $\angle$AGB contains m,
and the $\angle$CHD contains n;

$\therefore$ the $\angle$AGB : the $\angle$CHD = $m : n$.

Hence the $\angle$AGB : the $\angle$CHD = the arc AB : the arc CD.

And since the $\angle$AEB = one half of the $\angle$AGB; *Theor.* 38.
and the $\angle$CFD = one half of the $\angle$CHD;

$\therefore$ the $\angle$AEB : the $\angle$CFD = the arc AB : the arc CD.

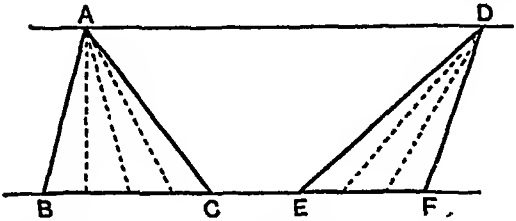
Q.E.D.

COROLLARY. Since *in equal circles, sectors which have equal angles are equal* [Theor. 42, Cor.], it may be proved as above that *the sector* AGB : *the sector* CHD = *the arc* AB : *the arc* CD.

PROPORTION APPLIED TO AREAS.

√ THEOREM 70. [Euclid VI. 1.]

The areas of triangles of equal altitude are to one another as their bases.



Let ABC, DEF be two triangles of equal altitude, standing on the bases BC, EF.

It is required to prove that

the △ ABC : *the* △ DEF = BC : EF.

Proof. Let the triangles be placed so that the bases BC, EF are in the same st. line, and the triangles on the same side of the line.

Join AD; then AD is par[l] to BF. *Def.* 2. p. 99.

Suppose the base BC : the base EF = $m : n$;
so that, if BC is divided into m equal parts, then EF may be divided into n such equal parts.

In each triangle let st. lines be drawn from the vertex to the points of division in the bases BC, EF.

Then the △[s] ABC, DEF are divided into triangles which stand on equal bases, and have the same altitude, and are therefore all equal.

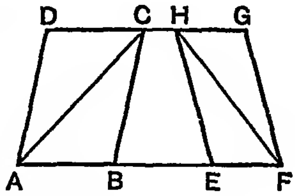
And of these equal △[s], the △ ABC contains m;
and the △ DEF contains n.

∴ the △ ABC : the △ DEF = $m : n$.

Hence the △ ABC : the △ DEF = BC : EF.

Q.E.D.

COROLLARY. *The areas of parallelograms of equal altitude are to one another as their bases.*



For let DB, EG be par^ms^ of the same altitude, standing on the bases AB, EF.

Join AC, HF.

Then

since the par^m^ DB = twice the $\triangle$CAB;

and the par^m^ EG = twice the $\triangle$HEF;

$\therefore$ the par^m^ DB : the par^m^ EG = the $\triangle$CAB : the $\triangle$HEF

= AB : EF.

ALTERNATIVE PROOF OF THEOREM 70.

Let p represent the altitude of each of the $\triangle$^s^ ABC, DEF.

Then the area of the $\triangle$ABC $= \frac{1}{2}$. base $\times$ altitude $= \frac{1}{2} . \text{BC} \times p$;

and the area of the $\triangle$DEF = ... $= \frac{1}{2} . \text{EF} \times p$.

$$\therefore \frac{\triangle \text{ABC}}{\triangle \text{DEF}} = \frac{\frac{1}{2} . \text{BC} \times p}{\frac{1}{2} . \text{EF} \times p} = \frac{\text{BC}}{\text{EF}}.$$

EXERCISES.

(Numerical.)

1. Two triangles of equal altitude stand on bases of 6·3″ and 5·4″ respectively; if the area of the first triangle is $12\frac{1}{4}$ square inches, find the area of the second.

2. The areas of two triangles of equal altitude have the ratio 24 : 17; if the base of the first is 4·2 cm., find the base of the second to the nearest millimetre.

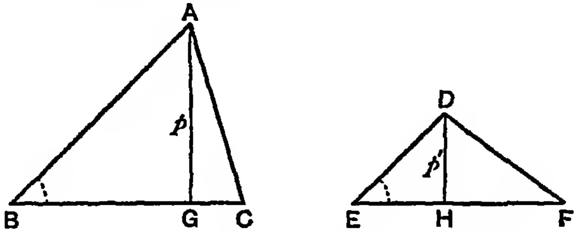
3. Two triangles lying between the same parallels have bases of 16·20 metres and 20·70 metres; find to the nearest square centimetre the area of the second triangle, if that of the first is 50·1204 sq. metres.

4. Two parallelograms whose areas are in the ratio 2·1 : 3·5 lie between the same parallels. If the base of the first is 6·6″ in length, find the base of the second.

5. Two triangular fields lie on opposite sides of a common base; and their altitudes with respect to it are 4·20 chains and 3·71 chains. If the first field contains 18 acres, find the acreage of the whole quadrilateral.

THEOREM 71.

If two triangles have one angle of the one equal to one angle of the other, their areas are proportional to the rectangles contained by the sides about the equal angles.



In the $\triangle^s$ ABC, DEF, let the $\angle^s$ at B and E be equal.

It is required to prove that

the $\triangle$ ABC : the $\triangle$ DEF = AB . BC : DE . EF.

Let AG and DH be drawn perp. to BC, EF respectively, and denote the lengths of these perp^s by p and p'.

Proof. The $\triangle$ ABC $= \frac{1}{2}$BC . p; and the $\triangle$ DEF $= \frac{1}{2}$EF . p'

$$\therefore \frac{\triangle ABC}{\triangle DEF} = \frac{BC \cdot p}{EF \cdot p'} \text{..........................(i)}$$

But since the $\angle$ B = the $\angle$ E, and the $\angle$ G = the $\angle$ H,

the $\triangle^s$ ABG, DEH are equiangular to one another; *Theor.* 16

$$\therefore \frac{p}{p'} = \frac{AB}{DE} \text{..............(ii)} \quad \textit{Theor. } 62$$

Substituting for $\frac{p}{p'}$ in (i),

$$\frac{\triangle ABC}{\triangle DEF} = \frac{BC \cdot AB}{EF \cdot DE};$$

or the $\triangle$ ABC : the $\triangle$ DEF = AB . BC : DE . EF.

Q.E.D.

COROLLARY. *Similarly it may be shewn that* parallelograms *having one angle of the one equal to one angle of the other are proportional to the rectangles contained by the sides about the equal angles.*

EXERCISES ON AREAS.

(*On Theorem* 70.)

1. √Assuming *the area of a triangle* $=\frac{1}{2}$ *base* $\times$ *altitude*, prove that *triangles on equal bases are proportional to their altitudes.*

Also deduce this result geometrically from Theorem 70.

2. XY is drawn parallel to BC, the base of the triangle ABC, cutting the sides AB, AC in X and Y.

Join BY and CX, and prove, by Theorem 70, that

(i) $AX : XB = AY : YC$.

(ii) $AB : AX = AC : AY$.

3. Shew that every quadrilateral is divided by its diagonals into four triangles whose areas are proportionals.

4. If two triangles are on equal bases and between the same parallels, any straight line parallel to their bases will cut off equal areas from the two triangles.

(*On Theorem* 71.)

5. In two triangles ABC, DEF, the $\angle B =$ the $\angle E$. If AB, BC are 2·7″ and 3·5″ respectively, and DE, EF are 2·1″ and 1·8″, shew that

$$\triangle ABC : \triangle DEF = 5 : 2.$$

6. The $\triangle^s$ ABC, DEF are equal in area, and the $\angle B =$ the $\angle E$. If AB = 5·6 cm., BC = 6·3 cm., DE = 7·2 cm., find EF.

7. In two parallelograms ABCD, EFGH, the $\angle B =$ the $\angle F$, and the areas have the ratio 3 : 4. If AB = 4·8 cm., BC = 13·5 cm., EF = 10·8 cm., find FG.

If p and p' denote the perpendiculars drawn from A and E to BC and FG respectively, shew that $p : p' = 4 : 9$.

8. Prove the formula

$$\textit{area of } \triangle = \tfrac{1}{2}\, ab \sin C\,;$$

and deduce Theorem 71.

9. The $\triangle ABC =$ the $\triangle DEF$ in area: and $AB : DE = EF : BC$; shew that the $\angle^s$ B and E are equal or supplementary.

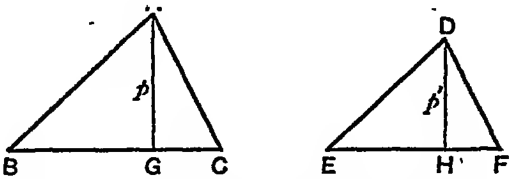
10. The sides AB, AC of the triangle ABC are cut by any straight line at P and Q respectively. By joining PC, and twice applying Theorem 70, shew that

$$\triangle APQ : \triangle ABC = AP \cdot AQ : AB \cdot AC.$$

Hence obtain an alternative proof of Theorem 71.

Theorem 72. [Euclid VI. 19.]

The areas of similar triangles are proportional to the squares on corresponding sides.



Let ABC, DEF be similar triangles, in which BC and EF are corresponding sides.

It is required to prove that

the $\triangle$ABC : the $\triangle$DEF = $BC^2 : EF^2$.

Let AG and DH be drawn perp. to BC, EF respectively; and denote these perp^s by p and p'.

Proof. The $\triangle ABC = \frac{1}{2}BC \cdot p$; and the $\triangle DEF = \frac{1}{2}EF \cdot p'$.

$$\therefore \frac{\triangle ABC}{\triangle DEF} = \frac{BC \cdot p}{EF \cdot p'} \text{..........................(i).}$$

But since the $\angle$B = the $\angle$E, from the similar $\triangle^s$ ABC, DEF,

and the $\angle$G = the $\angle$H, being right angles;

$\therefore$ the $\triangle^s$ ABG, DEH are equiangular to one another; *Theor.* 16.

$$\therefore \frac{p}{p'} = \frac{AB}{DE} \qquad \textit{Theor. } 62.$$

$$= \frac{BC}{EF}, \text{ from the similar } \triangle^s \text{ ABC, DEF.}$$

Substituting for $\frac{p}{p'}$ in (i),

$$\frac{\triangle ABC}{\triangle DEF} = \frac{BC \cdot BC}{EF \cdot EF} = \frac{BC^2}{EF^2};$$

or, the $\triangle$ABC : the $\triangle$DEF = $BC^2 : EF^2$.

Q.E.D.

EXERCISES ON THE AREAS OF SIMILAR TRIANGLES.

(Numerical and Graphical.)

1. In any triangle ABC, the sides AB, AC are cut by a line XY drawn parallel to BC. If AX is one-third of AB, what part is the triangle AXY of the triangle ABC?

2. Two corresponding sides of similar triangles are 3 ft. 6 in. and 2 ft. 4 in. respectively. If the area of the greater triangle is 45 sq. ft., find that of the smaller.

3. The area of the triangle ABC is 25·6 sq. cm., and XY, drawn parallel to BC, cuts AB in the ratio 5 : 3. Find the area of the triangle AXY.

4. Two similar triangles have areas of 392 sq. cm. and 200 sq. cm. respectively; find the ratio of any pair of corresponding sides.

5. ABC and XYZ are two similar triangles whose areas are respectively 32 sq. in. and 60·5 sq. in. If XY=7·7″, find the length of the corresponding side AB.

6. Shew how to draw a straight line XY parallel to BC the base of a triangle ABC, so that the area of the triangle AXY may be nine-sixteenths of that of the triangle ABC.

(Theoretical.)

7. ABC is a triangle, right-angled at A, and AD is drawn perpendicular to BC, shew that

$$\triangle BAD : \triangle ACD = BA^2 : AC^2.$$

8. A trapezium ABCD has its sides AB, CD parallel, and its diagonals intersect at O. If AB is double of CD, find the ratio of the triangle AOB to the triangle COD.

9. XY is drawn parallel to BC the base of the triangle ABC, if

$$\triangle AXY : \text{fig. } XBCY = 4 : 5,$$

shew that

$$AX : XB = 2 : 1.$$

10. Prove that the areas of similar triangles have the same ratio as the *squares of*

(i) corresponding altitudes;

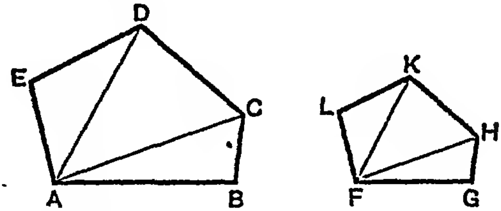
(ii) corresponding medians;

(iii) the radii of their in-circles;

(iv) the radii of their circum-circles.

THEOREM 73. [Euclid VI. 20.]

The areas of similar polygons are proportional to the squares on corresponding sides.



Let ABCDE, FGHKL be similar polygons, and let AB, FG be corresponding sides.

It is required to prove that

the polygon ABCDE : *the polygon* FGHKL $=$ AB^2 : FG^2.

Join AC, AD, FH, FK.

Proof. Then the $\triangle^s$ ABC, FGH are similar; *Theor.* 67.

also the $\triangle^s$ ACD, FHK are similar;

and the $\triangle^s$ ADE, FKL are similar.

$\therefore$ the $\triangle$ ABC : the $\triangle$ FGH $=$ AC^2 : FH^2 *Theor.* 72.

$=$ the $\triangle$ ACD : the $\triangle$ FHK.

Similarly,

the $\triangle$ ACD : the $\triangle$ FHK $=$ AD^2 : FK^2

$=$ the $\triangle$ ADE : the $\triangle$ FKL.

Hence $$\frac{\triangle ABC}{\triangle FGH} = \frac{\triangle ACD}{\triangle FHK} = \frac{\triangle ADE}{\triangle FKL}.$$

And in this series of equal ratios, the sum of the antecedents is to the sum of the consequents as each antecedent is to its consequent; *Theor.* V. p. 251.

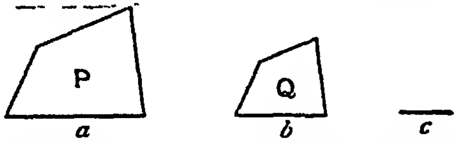
$\therefore$ the fig. ABCDE : the fig. FGHKL $=$ the $\triangle$ ABC : the $\triangle$ FGH

$=$ AB^2 : FG^2.

Q.E.D.

COROLLARY 1. Let a, b, c represent three lines in proportion, so that

$$\frac{a}{b}=\frac{b}{c}; \text{ and consequently } b^2=ac.$$



Now suppose similar figures P and Q to be drawn on a and b as corresponding sides,

then
$$\frac{\text{Fig. P}}{\text{Fig. Q}}=\frac{a^2}{b^2}=\frac{a^2}{ac}=\frac{a}{c}.$$

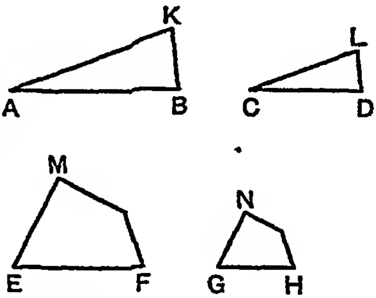
Hence if three straight lines are proportionals, and any similar figures are drawn on the first and second as corresponding sides, then

the fig. on the first : *the fig. on the* second = *the* first : *the* third.

COROLLARY 2. Let

AB : CD = EF : GH ;

and let similar figures KAB, LCD be similarly described on AB, CD, and also let similar figures MF, NH be similarly described on EF, GH.



Then since $\frac{AB}{CD}=\frac{EF}{GH}$; $\therefore \frac{AB^2}{CD^2}=\frac{EF^2}{GH^2}$.

But the fig. KAB : the fig. LCD $=AB^2 : CD^2$; *Theor.* 73.

and the fig. MF : the fig. NH $=EF^2 : GH^2$.

$\therefore$ the fig. KAB : the fig. LCD = the fig. MF : the fig. NH.

Hence if four straight lines are proportional, and a pair of similar rectilineal figures are similarly described on the first and second, and also a pair on the third and fourth, these figures are proportional.

EXERCISES ON THE AREAS OF SIMILAR FIGURES.

(Numerical and Graphical.)

1. Shew how to draw a straight line XY parallel to the base BC of a triangle ABC, so that the area of the triangle AXY may be four-ninths of the triangle ABC.

2. The sides of a triangle are 2·0", 2·5", 3·2"; find the sides of a similar triangle of three times the area.

[The results are to be given to the nearest hundredth of an inch.]

3. Two similar triangles have areas in the ratio 13·69 : 16·81, and an altitude of the greater is 10 ft. 3 in. Find the corresponding altitude of the other.

4. ABC is a triangle whose area is 16 sq. cm.; and XY is drawn parallel to BC, dividing AB in the ratio 3 : 5; if BY is joined, find the area of the triangle BXY.

5. One-fifth of the area of the triangle ABC is cut off by a line XY drawn parallel to BC. If BC = 10 cm., find XY to the nearest millimetre.

6. The area of a regular pentagon on a side of 2·5" is approximately 10¾ sq. in.; find the area of a similar figure on a side of 3·0".

7. The length of a rectangular area is 10·8 metres, and the ratio of the length to the breadth is 12 : 5. Find the length and breadth of a similar rectangle containing one-ninth of the area.

8. In the plan of a certain field, 1" represents 66 yards; if the area of the plan is found to be 100 sq. in., find the area of the field in acres.

Explain why in this example the *shape* of the field is immaterial.

9. An estate is represented on a plan by a quadrilateral ABCD drawn to the scale of 25" to the mile. If AC = 20", and the offsets from AC to B and D measure 24" and 26" respectively, find the acreage of the estate.

10. A field of 1·89 hectares is represented on a plan by a triangle whose sides measure 13 cm., 14 cm., and 15 cm. On what scale is the plan drawn?

EXERCISES ON THE AREAS OF SIMILAR FIGURES

(Theoretical.)

1. If ABC is a triangle, right-angled at A, and AD is drawn perpendicular to BC, shew that

(i) $BC^2 : BA^2 = BC : BD$; [Theor. 73, Cor. 1.]

(ii) $BC^2 : CA^2 = BC : CD$.

Hence deduce $BC^2 = BA^2 + AC^2$.

2. A triangle ABC is bisected by a straight line XY drawn parallel to the base BC. Determine the ratio AX : AB.

Hence shew how to bisect a triangle by a straight line drawn parallel to the base.

3. Two circles have external contact at A, and a common tangent, touching them at B and C, meets the line of centres at S. If AB, AC are joined, shew that

$$\triangle SBA : \triangle SAC = SB : SC.$$

4. Two circles intersect at A and B, and at A tangents are drawn, one to each circle, meeting the circumferences at C and D. If AB, CB and BD are joined, shew that

$$\triangle CBA : \triangle ABD = CB : BD.$$

5. DEF is the pedal triangle [see p. 207] of the triangle ABC; prove that $\triangle ABC : \triangle DBF = AB^2 : BD^2$.

Hence shew that

$$\text{fig. } AFDC : \triangle DBF = AD^2 : BD^2.$$

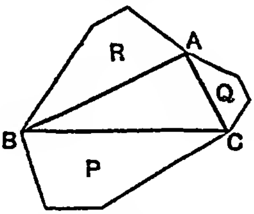
6. In a given triangle ABC a second triangle is inscribed by joining the middle points of the sides. In this inscribed triangle a third is inscribed in like manner, and so on. What fraction is the fourth triangle of the triangle ABC?

7. A regular hexagon is drawn on a side of a centimetres, and a second hexagon is inscribed in it by joining the middle points of the sides in order. In like manner a third hexagon is inscribed in the second, and so on. Find the ratio of the first hexagon to the fifth.

8. *Shew that the areas of two similar cyclic figures are proportional to the squares of the diameters of their circum-circles.* [Euclid XII. 1.]

THEOREM 74. [Euclid VI. 31.]

In a right-angled triangle, any rectilineal figure described on the hypotenuse is equal to the sum of the two similar and similarly described figures on the sides containing the right angle.



Let ABC be a right-angled triangle of which BC is the hypotenuse; and let P, Q, R be similar and similarly described figures on BC, CA, AB respectively.

It is required to prove that

the fig. R + the fig. Q = the fig P.

Proof. Since AB and BC are corresponding sides of the similar figs. R and P,

$$\therefore \frac{\text{fig. R}}{\text{fig. P}} = \frac{AB^2}{BC^2} \text{..........(i)} \quad \textit{Theor. 73.}$$

In like manner,

$$\frac{\text{fig. Q}}{\text{fig. P}} = \frac{AC^2}{BC^2} \text{..........(ii)}$$

Adding the equal ratios on each side in (i) and (ii)

$$\frac{\text{fig. R} + \text{fig. Q}}{\text{fig. P}} = \frac{AB^2 + AC^2}{BC^2}.$$

But $$AB^2 + AC^2 = BC^2; \qquad \textit{Theor. 29.}$$

$\therefore$ the fig. R + the fig. Q = the fig. P Q.E.D.

COROLLARY. *The area of a circle drawn on the hypotenuse of a right-angled triangle as diameter is equal to the sum of the circles similarly drawn on the other sides.*

For the areas of circles are proportional to the squares on their diameters. [Page 203.]

EXERCISES.

(*Miscellaneous.*)

1. In a triangle ABC, right-angled at A, AD is drawn perpendicular to the hypotenuse. Shew that

$$\text{(i) } BA^2 = BC.BD; \quad \text{(ii) } CA^2 = CB.CD.$$

Hence deduce Theorem 29, namely,

$$BC^2 = BA^2 + AC^2.$$

2. In the diagram of Theorem 74, draw AD perpendicular to BC; hence prove that, if the fig. F = the △ABC, then

(i) the fig. Q = the △ADC; (ii) the fig. R = the △ADB.

3. In the diagram of Theorem 74, if AB : AC = 8 : 5, and if the fig. P = 8·9 sq. cm., find the areas of the figs. Q and R.

4. BY and CZ are medians of the triangle ABC, and YZ is joined. Find the ratio of the triangle BGC to the triangle YGZ. [See p. 97.]

5. ABC is an isosceles triangle, the equal sides AB, AC each measuring 3·6". From a point D in AB, a straight line DE is drawn cutting AC produced at E, and making the triangle ADE equal in area to the triangle ABC. If AD = 1·8", find AE.

6. AB is a diameter of a circle, and two chords AP, AQ are produced to meet the tangent at B in X and Y.

Shew that (i) the △s APQ, AYX are similar;

(ii) the four points P, Q, Y, X are concyclic.

7. In the triangle ABC, the angle A is externally bisected by a line which meets the base produced at D and the circum-circle at E: shew that

$$AB.AC = AE.AD.$$

8. When is a straight line said to be divided *in extreme and mean ratio*?

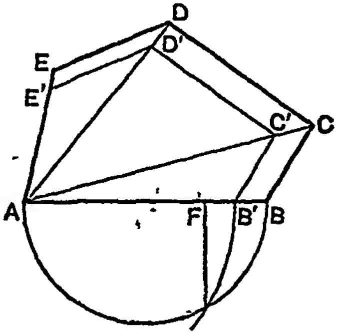
If a line 10 cm. in length is so divided, find the approximate lengths of the segments, and check your work graphically.

9. Draw an isosceles triangle equal in area to a triangle ABC, and having its vertical angle equal to the angle A.

10. On a given base draw an isosceles triangle equal in area to a given triangle ABC.

PROBLEM 40.

To draw a figure similar to a given rectilineal figure, and equal to a given fraction of it in area.



Let ABCDE be the given figure, to which a similar figure is to be drawn, having its area a given fraction (say *three-fourths*) of that of the fig. ABCDE.

Construction. Make AF *three-fourths* of AB. *Prob.* 7.

From AB cut off AB′ the mean proportional between AF and AB. *Prob.* 38 *Note.*

On AB′ draw the fig. AB′C′D′E′ similar to the fig. ABCDE. *Prob.* 39.

Then the fig. AB′C′D′E′ $= \frac{3}{4}$ of the fig. ABCDE.

Proof. By construction, $AB'^2 = AF \cdot AB$.

Now the figs. ABCDE, AB′C′D′E′ are similar, and AB, AB′ are corresponding sides;

$$\therefore \frac{\text{fig. } AB'C'D'E'}{\text{fig. } ABCDE} = \frac{AB'^2}{AB^2} \qquad \textit{Theor. 71}$$

$$= \frac{AF \cdot AB}{AB^2}$$

$$= \frac{AF}{AB} = \frac{3}{4}.$$

EXERCISES.

1. Divide a triangle ABC into two parts of equal area by a line XY drawn parallel to the base BC and cutting the other sides at X and Y.

Find (i) by calculation, (ii) by measurement, the ratio AX : AB.

2. Divide a triangle ABC into three parts of equal area by lines PQ, XY drawn parallel to the base BC. If P and X lie on AB, prove that

$$\frac{AP}{1}=\frac{AX}{\sqrt{2}}=\frac{AB}{\sqrt{3}}.$$

Hence shew how a triangle may be divided into n equal parts by lines drawn parallel to one side.

3. Draw a rectangle of length 8 cm., and breadth 5 cm. Then draw a similar rectangle of one-third the area.

Measure its length to the nearest millimetre, and verify your result by calculation.

4. Draw a quadrilateral ABCD from the following data:

the $\angle A=90^\circ$; AB=BC=8 cm.; AD=DC=6 cm.

Draw a similar quadrilateral to contain an area of 36 sq. cm., and find to the nearest millimetre the length of the side corresponding to AB.

5. Divide a circle of radius 3" into three equal parts by means of two concentric circles.

6. *Draw a rectilineal figure equal in area to a given figure* E, *and similar to a given figure* S. [Euclid VI. 25.]

[First replace the given figures E and S by equivalent squares (see Problems 19 and 32). Let the sides of these squares be a and b respectively, and let s be one of the sides of S.

Find p, a fourth proportional, to b, a, s, so that $b : a = s : p$

On p draw a figure P similar to the figure S, so that p and s are corresponding sides. Then P is the figure required;

for $$\frac{P}{S}=\frac{p^2}{s^2}=\frac{a^2}{b^2}=\frac{E}{S}.$$

$\therefore$ the fig. P=the fig. E.]

RECTANGLES IN CONNECTION WITH CIRCLES.

NOTE. We here give a simple proof of Theorems 57 and 58 brought under a single enunciation. [See NOTE p. 234.]

THEOREM 75. [Euclid III. 35 and 36.]

If any two chords of a circle cut one another internally or externally, the rectangle contained by the segments of one is equal to the rectangle contained by the segments of the other.

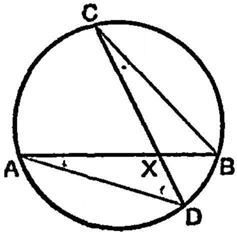


Fig. 1.

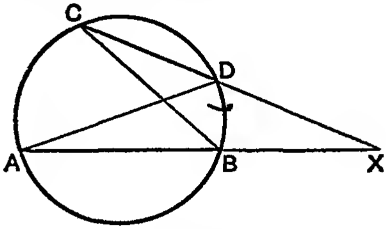


Fig. 2.

In the ⊙ ABC, let the chords AB, CD cut one another at X, internally in Fig. 1, and externally in Fig. 2.

It is required to prove in both cases that

the rect. XA, XB = *the rect.* XC, XD.

Join AD, BC

Proof. In the $\triangle^s$ AXD, CXB,

the ∠AXD = the ∠CXB, being opp. vert. $\angle^s$ in Fig. 1, and the same angle in Fig 2;

and the ∠A = the ∠C, being $\angle^s$ at the $\bigcirc^{ce}$, standing on the same arc BD;

∴ the remaining angles are equal; *Theor.* 16.

hence the $\triangle^s$ AXD, CXB are equiangular,

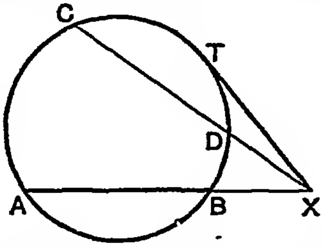
$$\therefore \frac{XA}{XC} = \frac{XD}{XB};$$

$$\therefore XA \cdot XB = XC \cdot XD:$$

that is, the rect. XA, XB = the rect. XC, XD.

Q.E.D.

COROLLARY. *If from an external point a secant and a tangent are drawn to a circle, the rectangle contained by the whole secant and the part of it outside the circle is equal to the square on the tangent.*



Let XBA be a secant, and XT a tangent drawn to the ⊙ABC from the point X.

It is required to prove that $XA \cdot XB = XT^2$.

Let XDC be a second secant;

then $XA \cdot XB = XC \cdot XD$, *Theor.* 75. Fig. 2.

and this is true for all positions of the line XDC.

Now let XDC turn about X away from the centre, so that the points C and D continually approach one another and ultimately coincide at T;

then XDC becomes the tangent XT,

and $XC \cdot XD$ becomes $XT \cdot XT$, or XT^2,

$\therefore$, ultimately, $XA \cdot XB = XT^2$.

EXERCISES FOR SQUARED PAPER.

1. From the point (1·7, 0) as centre, a circle is drawn to touch OY at O, and cutting OX at A. If *any* line is drawn from A to cut OY at Q and the circle at P, shew that $AP \cdot AQ$ is constant, and find its value when 1″ is taken as the unit of length.

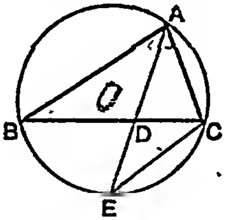
2. A circle of radius 10 is drawn from centre C (5, 6). If TT′ is the chord of contact of tangents from P (29, 16), and if PC meets TT′ in Q find the value of

(i) $CQ \cdot CP$; (ii) $PQ \cdot CP$; and (iii) the length of TT′.

3. From centres (·3, 0) (2, 0) circles of radii 2·6 and 2·5 respectively are drawn. Find the coordinates of their common points, and the length of their common chord. Also find the length of a tangent to each circle from the point (1·3, 3·4). Verify your results by measurement.

* THEOREM 76.

If the vertical angle of a triangle is bisected by a straight line which cuts the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base, together with the square on the straight line which bisects the angle.



Let ABC be a triangle, having the ∠ BAC bisected by AD.

It is required to prove that

the rect. AB, AC = *the rect.* BD, DC + *the sq. on* AD.

Suppose a circle circumscribed about the △ ABC; and let AD be produced to meet the $\odot^{ce}$ at E.

Join EC.

Proof. Then in the △ˢ BAD, EAC,

because the ∠ BAD = the ∠ EAC,

and the ∠ ABD = the ∠ AEC in the same segment;

∴ the remaining ∠ BDA = the remaining ∠ ECA;

that is, the △ˢ BAD, EAC are equiangular to one another;

$$\therefore \frac{AB}{AE} = \frac{AD}{AC}. \qquad \textit{Theor. } 62.$$

Hence $AB \,.\, AC = AE \,.\, AD$

$= (AD + DE)\, AD$

$= AD^2 + AD \,.\, DE.$

But $AD \,.\, DE = BD \,.\, DC$; *Theor.* 75.

∴ the rect. AB, AC = the rect. BD, DC + the sq. on AD.

Q.E.D.

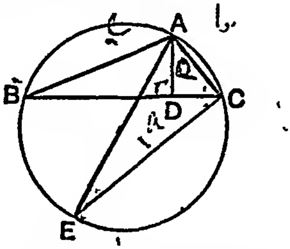
EXERCISE.

If the vertical angle BAC is bisected *externally* by AD, shew that

$AB \,.\, AC = BD \,.\, DC - AD^2$

*Theorem 77.

If from the vertical angle of a triangle a straight line is drawn perpendicular to the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circum-circle.



In the $\triangle$ABC, let AD be the perp. from A to the base BC: and let AE be a diameter of the circum-circle.

It is required to prove that

the rect. AB, AC = *the rect.* AE, AD.

Join EC.

Proof. Then in the $\triangle$^s BAD, EAC,

the rt angle BDA = the rt. angle ECA, in the semicircle ECA,

and the $\angle$ABD = the $\angle$AEC, in the same segment;

$\therefore$ the remaining $\angle$BAD = the remaining $\angle$EAC;

that is, the $\triangle$^s BAD, EAC are equiangular to one another.

$$\therefore \frac{AB}{AE} = \frac{AD}{AC}; \qquad \textit{Theor. 62.}$$

Hence $$AB \cdot AC = AE \cdot AD;$$

or the rect. AB, AC = the rect. AE, AD.

Q.E.D.

Note. Let a, b, c denote the sides of the $\triangle$ABC, R its circum-radius, and p the perp. AD.

Then since $$AE \cdot AD = AB \cdot AC$$

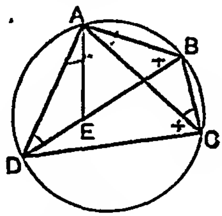
$$2R \cdot p = cb$$

$$\therefore R = \frac{bc}{2p}$$

$$= \frac{abc}{2ap} = \frac{abc}{4\Delta}.$$

Theorem 78. [Ptolemy's Theorem.]

The rectangle contained by the diagonals of a quadrilateral inscribed in a circle is equal to the sum of the two rectangles contained by its opposite sides.



Let ABCD be a quadrilateral inscribed in a circle, and let AC, BD be its diagonals.

It is required to prove that

the rect. AC, BD = *the rect.* AB, CD + *the rect.* BC, DA.

Make the ∠ DAE equal to the ∠ BAC;
to each add the ∠ EAC,
then the ∠ DAC = the ∠ EAB.

Proof. Then in the △ˢ EAB, DAC,
the ∠ EAB = the ∠ DAC,
and the ∠ ABE = the ∠ ACD in the same segment;
∴ the △ˢ EAB, DAC are equiangular to one another; *Theor.* 16.

$$\therefore \frac{BA}{CA} = \frac{BE}{CD}; \qquad \textit{Theor. } 62$$

hence AB . CD = AC . BE. .... (i)

Again in the △ˢ DAE, CAB,
the ∠ DAE = the ∠ CAB,
and the ∠ ADE = the ∠ ACB, in the same segment;
∴ the △ˢ DAE, CAB are equiangular to one another;

$$\therefore \frac{DA}{CA} = \frac{DE}{CB};$$

hence BC . DA = AC . DE. .... (ii)

Adding the equal rectangles on each side in (i) and (ii)

AB . CD + BC . DA = AC . BE + AC . DE
= AC (BE + DE)
= AC . BD.

Q.E.D.

EXERCISES.

1. ABC is an isosceles triangle, and on the base, or base produced, any point X is taken: shew that the circumscribed circles of the triangles ABX, ACX are equal.

2. From the extremities B, C of the base of an isosceles triangle ABC, straight lines are drawn perpendicular to AB, AC respectively, and intersecting at D: shew that

$$BC \, . \, AD = 2AB \, . \, DB.$$

3. If the diagonals of a quadrilateral inscribed in a circle are at right angles, the sum of the rectangles contained by the opposite sides is double the area of the figure.

4. ABCD is a quadrilateral inscribed in a circle, and the diagonal BD bisects AC: shew that

$$AD \, . \, AB = DC \, . \, CB.$$

5. If the vertex A of a triangle ABC is joined to any point in the base, it will divide the triangle into two triangles such that their circumscribed circles have radii in the ratio of AB to AC.

6. Construct a triangle, having given the base, the vertical angle, and the rectangle contained by the sides.

7. Two triangles of equal area are inscribed in the same circle: shew that the rectangle contained by any two sides of the one is to the rectangle contained by any two sides of the other as the base of the second is to the base of the first.

8. P is a point on the arc BC of the circum-circle of an equilateral triangle ABC. If P is joined to A, B, and C, shew that

$$PB + PC = PA.$$

9. ABCD is a quadrilateral inscribed in a circle, and BD bisects the angle ABC: if the points A and C are fixed on the circumference of the circle, and B is variable in position, shew that

$$AB + BC : BD \text{ is a constant ratio.}$$

10. From the formula $R = \dfrac{abc}{4\Delta}$ (see NOTE, p. 303) find the value of R when the sides of the triangle are as follows:

(i) 21″, 20″, 13″; (ii) 30 ft., 25 ft., 11 ft.

Draw the triangles to a convenient scale and check your work by measurement.

MISCELLANEOUS THEORETICAL EXAMPLES ON PARTS I.–V.

1. Two circles whose centres are C and D intersect at A and B; and a straight line PAQ is drawn through A and terminated by the circumferences: prove that

(i) the $\angle$PBQ = the $\angle$CAD;

(ii) the $\angle$BPC = the $\angle$BQD.

2. AB is a given diameter of a circle, and CD is any chord parallel to AB; if any point X in AB is joined to the extremities of CD, shew that

$$XC^2 + XD^2 = XA^2 + XB^2.$$

3. The opposite sides of a cyclic quadrilateral are produced to meet: shew that the bisectors of the two angles so formed are perpendicular to one another.

4. Given the vertical angle, one of the sides containing it, and the length of the perpendicular from the vertex on the base: construct the triangle.

5. A, B, C are three points in order in a straight line: find a point P in the straight line so that PB may be a mean proportional between PA and PC.

6. Through D, any point in the base of a triangle ABC, straight lines DE, DF are drawn parallel to the sides AB, AC, and meeting the sides at E, F: shew that the triangle AEF is a mean proportional between the triangles FBD, EDC.

---

7. PQ is a fixed chord in a circle, and PX, QY any two parallel chords through P and Q; shew that XY touches a fixed concentric circle.

8. Two circles touch each other at C, and straight lines are drawn through C at right angles to one another, meeting the circles at P, P′ and Q, Q′ respectively: if the straight line which joins the centres is terminated by the circumferences at A and A′ shew that

$$P'P^2 + Q'Q^2 = A'A^2.$$

9. AE bisects the vertical angle of the triangle ABC and meets the base in E. If d, d' are the diameters of the circum-circles of the triangles ABE, ACE, shew that

$$d : d' = BE : EC.$$

10. AB, AC are chords of a circle; a line parallel to the tangent at A cuts AB, AC in D and E respectively: shew that

$$AB \, . \, AD = AC \, . \, AE.$$

11. If a straight line is divided at two given points, determine a third point such that its distances from the extremities may be proportional to its distances from the given points.

12. Given the feet of the perpendiculars drawn from the vertices on the opposite sides: construct the triangle.

---

13. If a quadrilateral can have one circle inscribed in it, and another circumscribed about it, shew that the straight lines joining the opposite points of contact of the inscribed circle are perpendicular to one another.

14. Two equal circles move between two straight lines placed at right angles, so that each line is touched by one circle, and the two circles touch one another: find the locus of the point of contact.

15. AB is a diameter of a given circle; and AC, BD, two chords on the same side of AB intersect at E: shew that the circle which passes through D, E, C cuts the given circle orthogonally. [See Def. p. 330.]

16. If four circles are described to touch every three sides of a quadrilateral, shew that their centres are concyclic.

17. AB is a straight line divided at C and D so that

$$AB : AC = AC : AD;$$

from A a line AE is drawn in any direction and equal to AC; shew that BC and CD subtend equal angles at E.

18. Given the vertical angle, the ratio of the sides containing it, and the diameter of the circumscribing circle, construct the triangle.

---

19. O is a fixed point, and OP is any line drawn to meet a fixed straight line in P; if on OP a point Q is taken so that OQ to OP is a constant ratio, find the locus of Q.

20. O is a fixed point, and OP is any line drawn to meet the circumference of a fixed circle in P; if on OP a point Q is taken so that OQ to OP is a constant ratio, find the locus of Q.

21. Two equal circles intersect at A and B; and from C, any point on the circumference of one of them, a perpendicular is drawn to AB, meeting the other circle at O and O′; shew that either O or O′ is the orthocentre of the triangle ABC. Distinguish between the two cases.

22. Three equal circles pass through the same point A, and their other points of intersection are B, C, D: shew that of the four points A, B, C, D, each is the orthocentre of the triangle formed by joining the other three.

23. From a given point without a circle draw a straight line to the concave circumference so as to be bisected by the convex circumference. When is this problem impossible?

24. Given the base, the altitude, and the radius of the circum-circle: construct the triangle.

---

25. Given the base of a triangle and the sum of the remaining sides: find the locus of the foot of the perpendicular from one extremity of the base on the bisector of the exterior vertical angle.

26. Construct a triangle having given either the three ex-centres, or the in-centre and two ex-centres.

27. If O is the orthocentre of a triangle ABC, shew that

$$AO^2+BC^2=BO^2+CA^2=CO^2+AB^2=d^2,$$

where d is the diameter of the circum-circle.

28. If C is the middle point of an arc of a circle whose chord is AB, and D is any point in the conjugate arc; shew that

$$AD+DB : DC=AB : AC.$$

29. D is a point in the side AC of the triangle ABC, and E is a point in AB. If BD, CE divide each other into parts in the ratio 4 : 1, then D, E divide CA, BA in the ratio 3 : 1.

30. If the perpendiculars from two fixed points on a straight line passing between them are in a given ratio, the straight line must pass through a third fixed point.

---

31. From the vertex A of any triangle ABC draw a line meeting BC produced in D so that AD may be a mean proportional between the segments of the base.

32. Two circles touch internally at O: AB a chord of the larger circle touches the smaller in C which is cut by the lines OA, OB in the points P, Q: shew that OP : OQ=AC : CB.

33. AB is any chord of a circle; AC, BC are drawn to any point C in the circumference and meet the diameter perpendicular to AB at D, E: if O is the centre, shew that the rect. OD, OE is equal to the square on the radius.

34. YD is a tangent to a circle drawn from a point Y in the diameter AB produced; from D a perpendicular DX is drawn to the diameter; shew that the points X, Y divide AB internally and externally in the same ratio.

35. Determine a point in the circumference of a circle, from which lines drawn to two other given points shall have a given ratio.

36. Given the base, and the position of the bisector of the vertical angle: construct the triangle.

---

37. EA, EA′ are diameters of two circles touching each other externally at E; a chord AB of the former circle, when produced, touches the latter at C′, while a chord A′B′ of the latter touches the former at C: prove that

$$AB \cdot A'B' = 4BC' \cdot B'C.$$

38. From a given external point draw a straight line to cut off a quadrant from a given circle.

39. Shew that the straight lines joining the circum-centre of a triangle to its vertices are perpendicular to the corresponding sides of the pedal triangle.

40. P is any point on the circum-circle of a triangle ABC; and perpendiculars PD, PE are drawn to the sides BC, CA. Find the locus of the circum-centre of the triangle PDE.

41. P is any point on the circum-circle of a triangle ABC: shew that the angle between Simson's Line for the point P and the side BC is equal to the angle between AP and that diameter of the circum-circle which passes through A.

42. Given the base, the vertical angle, and the difference of the angles at the base: construct the triangle.

---

43. Shew that the circles circumscribed about the four triangles formed by two pairs of intersecting straight lines meet in a point.

44. Shew that the orthocentres of the four triangles formed by two pairs of intersecting straight lines are collinear.

45. Of all polygons of a given number of sides, which can be inscribed in a given circle, that which is regular has the maximum area and the maximum perimeter.

46. On a straight line PAB, two points A and B are marked and the line PAB is made to revolve round the fixed extremity P. C is a fixed point in the plane in which PAB revolves; prove that if CA and CB are joined, and the parallelogram CADB is completed, the locus of D will be a circle.

47. Describe an equilateral triangle equal to a given isosceles triangle.

48. Given the vertical angle of a triangle in position and magnitude, and the sum of the sides containing it: find the locus of the circum-centre.

---

49. ABC is any triangle, and on its sides equilateral triangles are described externally: if X, Y, Z are the centres of their in-circles shew that the triangle XYZ is equilateral.

50. In a given circle inscribe a triangle so that two sides may pass through two given points and the third side be parallel to a given straight line.

51. In a given circle inscribe a triangle so that the sides may pass through the three given points.

52. A, B, X, Y are four points in a straight line, and O is such a point in it that the rectangle OA, OY is equal to the rectangle OB, OX; if a circle is described with centre O and radius equal to a mean proportional between OA and OY, shew that at every point on this circle AB and XY will subtend equal angles.

53. Find the locus of a point which moves so that its distances from two intersecting straight lines are in a given ratio.

54. If S, I, I_1, are the circum-centre, in-centre, and an ex-centre of a triangle, and R, r, r_1 the radii of the corresponding circles, and if N is the centre of the nine-points circle, prove that

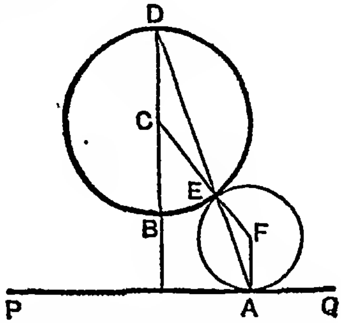
(i) $SI^2 = R^2 - 2Rr$; (ii) $SI_1^2 = R^2 + 2Rr_1$;

(iii) $NI = \frac{1}{2}R - r$; (iv) $NI_1 = \frac{1}{2}R + r_1$.

MISCELLANEOUS THEOREMS AND EXAMPLES.

I. SOME CONSTRUCTIONS OF CIRCLES.

EXAMPLE 1. *Draw a circle to touch a given circle* (C), *and also to touch a given straight line* PQ *at a given point* A.



Construction. At A draw AF perp. to PQ:

then the centre of the required $\odot$ must lie in AF.

Take C the centre of the given $\odot$, and draw the diam. BD perp. to PQ.

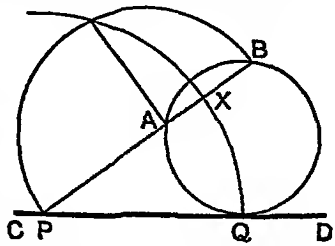
Join A to one extremity D, of the diameter, cutting the O^{ce} at E.

Join CE, and produce it to cut AF at F.

Then F *is the centre, and* FA *the radius of the required circle.*

[Supply the proof; and shew that a second solution is obtained by joining AB, and producing it to meet the O^{ce}.]

EXAMPLE 2. *Draw a circle to pass through two given points* A *and* B, *and to touch a given straight line* CD.



Construction. Join BA, and produce it to meet CD at P.

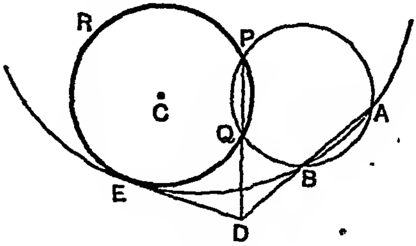
Find PX the mean proportional between PA and PB. *Prob.* 38, *Note.*

From PD (or PC) cut off PQ equal to PX.

Then the circle drawn through A, B, *and* Q [Prob. 25] *will touch* CD *at* Q.

[Supply the proof; and shew that there are in general two solutions. Modify the construction to meet the case when AB is parallel to CD.]

EXAMPLE 3. *Draw a circle to pass through two given points* A *and* B, *and to touch a given circle* (C).



Construction. Through A and B draw *any* circle to cut the given circle at P and Q.

Join AB, PQ and produce them to meet at D.

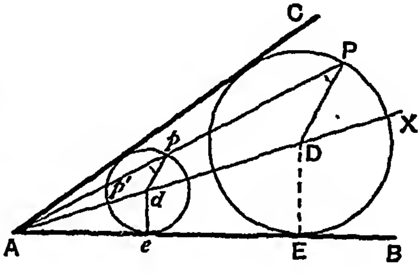
From D draw a tangent DE to the given circle.

Then the circle drawn through A, B, E *will touch the given circle at* E.

[Supply the proof from Theorems 58 and 59; and shew that there are in general two solutions.

Modify the construction to meet the case when the straight line bisecting AB at right angles passes through C.]

EXAMPLE 4. *Draw a circle to pass through a given point* P, *and to touch two given straight lines* AB, AC.



Construction. Draw AX bisecting the ∠ BAC. Then all circles touching AB, AC have their centres in AX.

From any point *d* in AX draw *de* perp. to AB; hence with *d* as centre, draw a circle touching AB and AC.

Join AP, cutting the ⊙ (*d*) at *p*, *p′*.

Join *pd*; and through P draw PD par[l] to *pd*, cutting AX at D.

Then D *is the centre, and* DP *the radius of the required circle touching* AB *and* AC.

[Draw DE perp. to AB. The proof is obtained by shewing that DE=DP, by means of the similar △s ADE, A*de*, and the similar △s ADP, A*dp*.

Shew that a second solution may be obtained by joining *dp′*, and proceeding as before.

Modify the construction to meet the case when the given lines are parallel.]

EXERCISES FOR SQUARED PAPER.

1. Given a circle of radius 10 having its centre at the origin, draw a circle to touch the given circle and also touch the x-axis at the point (20, 0).

Shew that two *equal* circles can be so drawn. Calculate the radius of that in the first quadrant and the coordinates of its point of contact with the given circle.

2. Given a circle of radius 10 having its centre at the origin, draw a circle to touch the given circle at the point (6, 8), and also to touch the y-axis.

Shew that two such circles can be drawn. Find their radii and points of contact with the y-axis.

3. Draw a quadrant of a circle of radius 2″, and inscribe a circle in it. Shew that the radius of the inscribed circle is the positive root of the equation $r^2+4r-4=0$.

Obtain the radius by calculation and by measurement.

4. Shew that two circles can be drawn to touch both axes of coordinates and to pass through the point (2″, 2″); and prove that their radii are given by the quadratic $r^2+4r\sqrt{2}-8=0$.

Draw the smaller of these circles and obtain its radius by measurement.

5. Join the points (2″, 0) and (0, 3″); also join the points (3″, 0) and (0, 2″); then draw a circle to touch the joining lines and to pass through the origin.

6. Within an equilateral triangle on a side of 3·0″ draw three equal circles each to touch two sides of the triangle and the other two circles.

If r is the radius of one of these circles, shew that

$$r(\tan 60^\circ+1)=\frac{3}{2}.$$

Hence find r to the nearest hundredth of an inch.

7. Within a circle of radius 2·0″ draw three equal circles each to touch the other two and the given circle.

If r is the radius of one of these equal circles shew that

$$r(1+\operatorname{cosec} 60^\circ)=2.$$

Hence find r to the nearest hundredth of an inch.

II. MAXIMA AND MINIMA.

When a line, angle, or figure varying under specified conditions, gradually changes its position and magnitude, we may be required to note if any situations exist in which, after increasing, it begins to decrease; or, after decreasing, to increase. In such situations the magnitude is said to have reached a maximum or minimum value. We propose here to deal with problems in which the variable magnitude admits of only *one* transition from an increasing to a decreasing state—and vice versâ: so that for our present purpose the maximum is actually the greatest, and the minimum actually the least value that the variable magnitude can take.

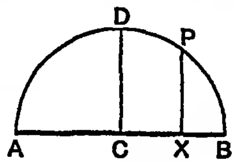
Two hints towards the solution of such problems may be given.

(i) Since a variable geometrical magnitude reaches its maximum or minimum value at a *turning point*, towards which the magnitude may mount or descend from either side, it is natural to expect a maximum or minimum value when the magnitude assumes a *symmetrical* form or position; and this is usually found to be the case.

EXAMPLE 1. *Divide a straight line* AB *internally so that the rectangle contained by the two segments may be a maximum.*

Bisect AB at C, and on AB draw a semi-circle.

Take *any* point X in AB; and draw XP perp. to AB to cut the O^{ce} at P.



Then $AX . XB = PX^2$. *Prob.* 32.

Now PX is greatest when it coincides with the radius CD;

$\therefore$ AX . XB is a maximum, when X is the mid-point of AB.

Observe that in this case the maximum is reached when PX occupies the *symmetrical position* in which it bisects AB at right angles.

(ii) Again we can find when a geometrical magnitude assumes its maximum or minimum value, if we can discover a construction for drawing the magnitude so that it may have an *assigned* value: for we may then examine between what limits the assigned value must lie in order that the construction may be possible; and the higher or lower limit will give the maximum or minimum sought for.

It has been pointed out that if under certain conditions existing among the data, *two* solutions of a problem are possible, and under other conditions, *no* solution exists, there will always be some intermediate condition under which the two solutions combine in a *single* solution. [See page 94.]

In these circumstances this single solution will be found to correspond to the maximum or minimum value of the magnitude to be constructed.

EXAMPLE 2. *To find at what point in* CD, *a given straight line of indefinite length, the angle subtended by a finite line* AB *is a maximum.*

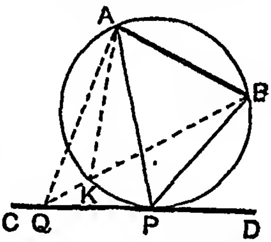
First find at what point in CD a *given* angle is subtended by AB.

This is done as follows:

On AB draw a segment of a circle containing an angle equal to the given angle. *Problem* 24.

If the arc of this segment intersects CD, *two* points in CD are found at which AB subtends the given angle: but if the arc does not meet CD, *no* solution is given.

In accordance with the principles explained above, we expect that the maximum angle is determined when the arc *touches* CD, that is, meets it at two coincident points. This we shall prove to be the case.



Draw a circle through A and B to touch CD; and let P be the point of contact. Ex. 2. p. 311.

Then the ∠APB *is greater than any other angle subtended by* AB *at a point in* CD *on the same side of* AB *as* P.

Proof. For take *any* other point Q in CD on the same side of AB as P; and join AQ, QB.

Let BQ meet the circle at K. Join AK.

Then the ∠AKB = the ∠APB, in the same segment.

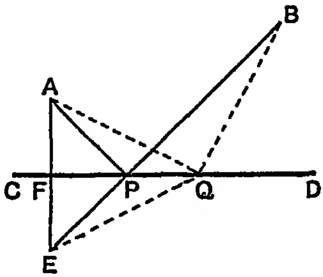
But the ext. ∠AKB is greater than the int. opp. ∠AQB;

∴ the ∠APB is greater than the ∠AQB.

Hence the ∠APB is a maximum.

NOTE. Two circles may be described to pass through A and B, and to touch CD, the points of contact being on opposite sides of AB; hence two points in CD may be found such that the angle subtended by AB at each of them is greater than the angle subtended at any other point in CD *on the same side of* AB.

EXAMPLE 3. *In* CD, *a straight line of indefinite length, find a point* P *such that the sum of its distances from two points* A, B *on the same side of* CD *is a minimum.*



Draw AF perp. to CD; and produce AF to E, making FE equal to AF.

Join EB, cutting CD at P. Join AP.

Then AP+PB *is a minimum.*

Proof. Take *any* other point Q in CD, and join AQ, BQ, EQ.

Now the Δ^s AFP, EFP are congruent,

∴ AP=EP.

Similarly AQ=EQ.

And in the Δ EQB, EQ+QB is greater than EB;

hence AQ+QB is greater than EB,

that is, greater than AP+PB.

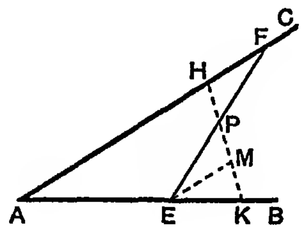
Thus AP+PB is a minimum.

NOTE. It follows that the ∠APF=the ∠EPF *Theor.* 4.

=the ∠BPD. *Theor.* 3.

Thus AP+BP is a minimum, when AP, PB are *equally inclined* to CD.

EXAMPLE 4. *Given two intersecting straight lines* AB, AC, *and a point* P *between them; shew that of all straight lines which pass through* P *and are terminated by* AB, AC, *that which is bisected at* P *cuts off the triangle of minimum area.*



Let EF be that st. line, terminated by AB, AC, which is bisected at P.

Then the Δ FAE *is of minimum area.*

Proof. For let HK be *any* other st. line passing through P.

Through E draw EM par[l] to AC.

Then the Δ^s HPF, MPE are evidently congruent, *Theor.* 17.

and are therefore equal in area.

∴ the Δ HPF is less than the Δ KPE.

To each add the fig. AHPE;

then the Δ FAE is less than the Δ HAK.

That is, the Δ FAE is a minimum.

EXERCISES ON MAXIMA AND MINIMA.

1. Two sides of a triangle are given in length; how must they be placed in order that the area of the triangle may be a maximum?

Find the area of the greatest triangle in which $a=6{\cdot}8$ cm., and $b=4{\cdot}5$ cm.

2. Of all triangles of given base and area, shew that that which is isosceles has the least perimeter. [See Ex. 3, p. 316.]

Calculate the minimum perimeter of a triangle of which the base$=2{\cdot}0''$, and the area$=3{\cdot}12$ sq. in.

3. Construct a triangle of maximum area on a base of 10 cm., and having a vertical angle of 60°. Calculate its area.

4. With the origin as centre draw a circle of radius $1{\cdot}5''$, and draw AB joining the points $(3'', 0)$, $(0, 3'')$. Find a point in AB such that the tangents drawn from it to the circle contain the maximum angle. Measure the angle, and account for the result.

5. A straight rod slides between two straight rulers placed at right angles to one another; in what position is the triangle intercepted between the rulers and rod a maximum?

6. Divide a given straight line into two parts, so that the sum of the squares on the segments

(i) may be equal to a given square;
(ii) may be a minimum.

7. Through a point of intersection of two circles draw a straight line terminated by the circumferences,

(i) so that it may be of given length;
(ii) so that it may be a maximum.

8. Draw a circle to touch the axes of x and y at two points A and B, each $2''$ distant from the origin.

Find a point on the major arc AB such that the sum of its coordinates is a maximum.

Also find a point on the minor arc AB such that the sum of its coordinates is a minimum.

In each case calculate the sum, and test by measurement.

9. Straight lines are drawn from two given points to meet one another on the convex circumference of a given circle: prove that their sum is a minimum when they make equal angles with the tangent at the point of intersection.

10. Shew that of all triangles having a given vertical angle and altitude, that which is isosceles has the least area.

What is the least area a triangle can have if its vertical angle$=60°$, and altitude$=6$ cm.? Find its perimeter.

11. Given two intersecting tangents to a circle, draw a tangent to the convex arc so that the triangle formed by it and the given tangents may be of maximum area.

12. Find graphically (to the nearest degree) the greatest vertical angle which a triangle may have, when its base $=1{\cdot}6''$, and its area $=1{\cdot}2$ sq. in.

13. A and B are two points both within, or both without, a given circle. Find a point on the circumference at which AB subtends the greatest angle. [See Ex. 2, p. 315.]

14. A and B are two points on the x-axis distant $0{\cdot}8''$ and $1{\cdot}8''$ from the origin O. Find graphically, a point P on the y-axis, such that the angle APB is a maximum.

Calculate the length of OP, and measure the maximum angle.

15. A bridge consists of three arches, whose spans are 49 ft., 32 ft. and 49 ft. respectively: how far from the bridge is the point on either bank of the river at which the middle arch subtends the greatest angle?

16. From a given point P without a circle whose centre is C, draw a straight line to cut the circumference at A and B, so that the triangle ACB may be of maximum area.

Find the area of the greatest triangle that can be so drawn, when the radius $=6$ cm., and shew that the area is independent of the position of P.

17. Find the area of the greatest rectangle which can be inscribed in a circle of radius $5{\cdot}5$ cm.

18. A and B are two fixed points without a circle: find a point P on the circumference, such that AP^2+PB^2 may be a minimum.

[See Theor. 56.]

19. A segment of a circle is described on the chord AB: find a point C on its arc so that the sum of AC, BC may be a maximum.

20. *Of all triangles that can be inscribed in a circle that which has the greatest perimeter is equilateral.*

21. *Of all triangles that can be inscribed in a given circle that which has the greatest area is equilateral.*

22. *Of all triangles that can be inscribed in a given triangle that which has the least perimeter is the pedal triangle.*

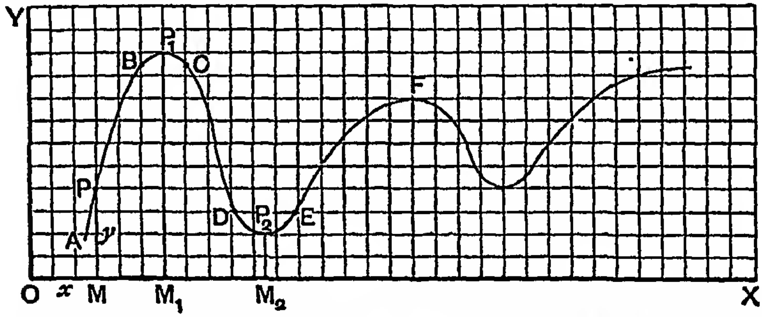
23. Of all rectangles of given area, the square has the least perimeter.

24. Describe the triangle of maximum area, having its angles equal to those of a given triangle, and its sides passing through three given points.

III. GRAPHS. APPLICATION TO MAXIMA AND MINIMA.

Problems dealing with the maximum or minimum values of some variable magnitude may often be conveniently treated by exhibiting its gradual changes by means of a graph. For details of graphical work the student may consult Hall's *Introduction to Graphical Algebra*. It will be sufficient here to explain the following general method of procedure. The variable magnitude whose values we have to examine may be denoted by y, and the quantity, in terms of which it is expressed, by x. By plotting a series of corresponding values of x and y on the coordinate axes OX, OY, a series of points is determined. If a continuous curve is drawn through them, the ordinate of each point denotes the value of the magnitude in question corresponding to a given value of the quantity x.

The advantage of this method is that it exhibits a visual picture of continuous change, so that the graph enables us to read off the value of y corresponding to *any* given value of x; and in particular the positions of maximum and minimum values are seen at a glance.



In this figure the continuous curve ABCDEF represents the graph of a variable quantity Q. As x increases gradually, the ordinate y travels parallel to OY, and its value at any point gives the value of Q for the corresponding value of x. At P_1 the value of y is greater than that at B or C on either side, and here Q is a maximum. Similarly at P_2 the value of y is less than that at D or E, and here Q is a minimum.

It will now be evident that maximum and minimum values occur at the *turning points* where the ordinates are algebraically greatest and least respectively in the immediate vicinity of such points.

The following points should also be noticed:

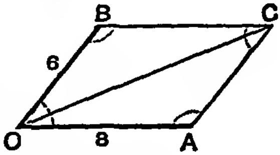
(i) In any continuous curve maximum and minimum values occur alternately.

(ii) There will always be a maximum or a minimum value between any two equal values of the ordinate.

(iii) The slope of the curve at any point indicates the rate of change at that point of the quantity under discussion, and at each point of maximum or minimum value the tangent to the curve is parallel to the axis of x.

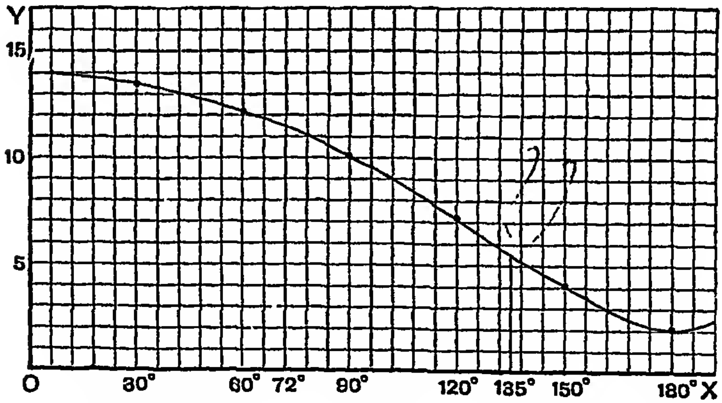
EXAMPLE 1. OACB *is a parallelogram in which* OA=8 cm. *and* OB=6 cm. *If* OB *rotates about* O, *trace the changes in the value of* OC *as the angle* AOB *increases from* 0° *to* 180°. *Illustrate the changes by means of a graph. Read off the value of* OC *for an angle of* 72°, *and find the value of the angle when* OC=5·6 cm.

By drawing a series of figures, increasing the angle AOB by increments of 30°, we shall find by measurement the corresponding values of OC and the ∠AOB to be as in the following table.



∠AOB	0°	30°	60°	90°	120°	150°	180°
OC	14·0	13·5	12·2	10·0	7·2	4·1	2·0

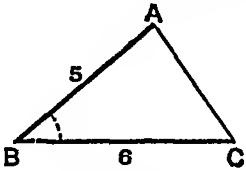
Denoting the angle by x, let its successive values be plotted on the x-axis, and let the corresponding values of OC be taken as ordinates. If each division on OX is taken to represent 6°, and each division on OY to represent 1 cm. we obtain the adjoining graph.



From this we see that when $x=72°$, $y=11·5$ cm., and when $y=5·6$ cm. $x=135°$.

The student should plot the graph for himself on a much larger scale than is possible on this page. He should also continue the values of OC for a few angles greater than 180°. He will then find that a minimum point has been reached when the angle AOB is 180°. Also it should be evident that at 360° the value of OC is again equal to 14, which is its maximum value.

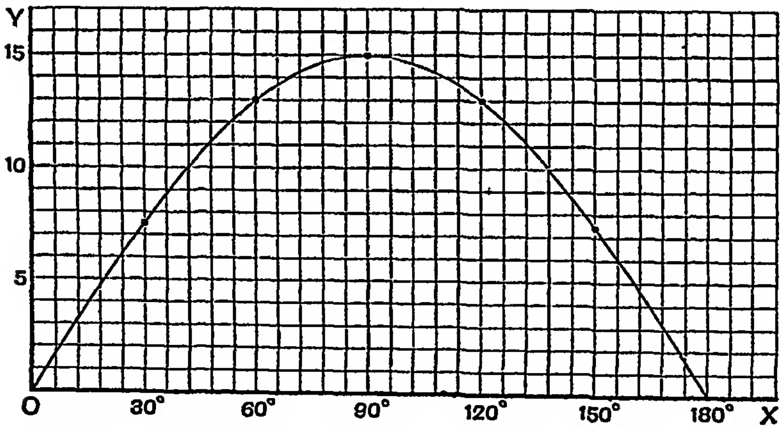
EXAMPLE 2. ABC *is a triangle in which* BC, BA *have constant lengths* 6 cm. and 5 cm. *If* BC *is fixed, and* BA *revolves about* B, *trace the changes in the area of the triangle as the angle* B *increases from* 0° *to* 180°. *Illustrate these changes by a graph, and determine for what values of the angle* B *the area is* 10 sq. cm. *Also find for what value of* B *the area is a maximum.*



Proceeding as in Ex. 5, p. 110, the corresponding values of the area and the angle will be as in the following table:

Angle	0°	30°	60°	90°	120°	150°	180°
Area in sq. cm.	0	7·5	13·0	15·0	13·0	7·5	0

Plot the values of the angle on the x-axis, and let the successive values of the area be taken as corresponding ordinates. Then with the same units as in the last example we obtain the adjoining graph.



It is easily seen that the maximum value of the ordinate is 15, corresponding to an angle of 90°.

Thus the area of the triangle is a maximum when the angle included by the given sides is a right angle.

Also the curve is symmetrical with regard to its maximum ordinate, so that there are two values of the angle which furnish a given area, other than the maximum. When the area is 10 sq. cm. the two values of the angle are 42° and 138°.

NOTE. The area of $\triangle ABC = \frac{1}{2} 5 . 6 \sin B = 15 \sin B$. Hence the graph may be plotted from a Table of sines. [See *Graphical Algebra*, p. 29.]

EXERCISES ON GRAPHS.

1. PQ is a perpendicular, 8 cm. in length, to a straight line XY, and PR is an oblique making an angle α with PQ. By giving to α the values 15°, 30°, 45°, 60°, 75°, find by measurement the corresponding values of PR, and tabulate the results. Illustrate the changes in PR by means of a graph, and find from it (i) the length of PR when $\alpha=63°$, (ii) the value of α when PR=8·8 cm.

2. In a triangle $a=4$ cm. $b=5$ cm. Plot a graph to shew the changes in the area of the triangle for different values of C. Find from the graph (i) the area of the triangle when C=63°; (ii) the *values* of C when the area is 9·5 sq. cm.; (iii) how the sides must be placed when the area is a maximum.

3. A straight rod AB of length 5 cm. slides between two straight rulers CD, CE placed at right angles to each other. Draw a graph to shew the variations of the area of the triangle BCA for different values of the length CA.

Point out the position of AB when the area is a maximum.

4. AB is a straight line 10 cm. in length divided internally at P. As P moves from A to B illustrate graphically the variations of

$$\text{(i) } AP \cdot PB; \quad \text{(ii) } AP^2+PB^2.$$

In each case determine from the graph the position of P which gives a maximum or minimum value.

5. In a triangle $c=6$ cm. and A=60°. Trace the changes of a graphically for different values of b. Find from the graph the minimum value of a. Draw the triangle for this value, and hence check your result.

6. Through A, the extremity of the diameter AB of a semi-circle, a line AP is drawn to the circumference. Trace graphically the variations of the area of the triangle BAP for different values of the angle PAB. Find the value of this angle when the area is greatest.

7. By means of Theorem 73, shew that the graph of the equation $y=mx^2$, where m is constant, exhibits the changes in area of any series of similar rectilineal figures similarly placed on sides of varying length. Draw a graph to shew the changes in the area of a square as its side varies, and from the graph find approximately the side of a square whose area is 11·8 sq. in.

8. Draw the graphs of the curves represented by

$$\text{(i) } y=2x-\frac{x^2}{4}; \quad \text{(ii) } y=5-4x-x^2$$

Find the maximum value of $5-4x-x^2$.

IV. HARMONIC SECTION.

DEFINITIONS.

1. Three quantities are said to be in **Arithmetical Progression** when the difference between the last pair is equal to that between the first pair.

Thus a, b, c are in A.P. when

$$c-b=b-a,$$

and b is said to be an **Arithmetic Mean** between a and c.

2. Three quantities are said to be in **Geometrical Progression** when the ratio of the third to the second is the same as that of the second to the first.

Thus a, b, c are in G.P. when

$$\frac{c}{b}=\frac{b}{a},$$

and b is said to be a **Geometric Mean** between a and c.

3. Three quantities are said to be in **Harmonical Progression** when the first bears to the third the same ratio as the difference between the first and second bears to the difference between the second and third.

Thus a, b, c are in H.P. when

$$\frac{a}{c}=\frac{a-b}{b-c},$$

and b is said to be a **Harmonic Mean** between a and c.

NOTE. Since, by definition,

$$\frac{b-c}{c}=\frac{a-b}{a}, \quad \text{or} \quad \frac{b-c}{bc}=\frac{a-b}{ab},$$

it follows that

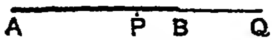
$$\frac{1}{c}-\frac{1}{b}=\frac{1}{b}-\frac{1}{a};$$

$\therefore$ the reciprocals $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P., a result which is often useful.

4. If A, G, H denote the arithmetic, geometric, and harmonic means respectively between two given quantities a and b, it easily follows from the above definitions that

$$A=\frac{a+b}{2}, \quad G=\sqrt{ab}, \quad H=\frac{2ab}{a+b}.$$

Definition. A finite straight line is said to be cut harmonically when it is divided internally and externally into segments which have the same ratio.



Thus AB is divided harmonically at P and Q, if

$$AP : PB = AQ : QB.$$

P and Q are said to be harmonic conjugates of A and B.

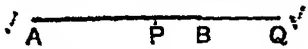
Now by taking the above proportion *alternately*,

we have $\qquad PA : AQ = PB : BQ;$

from which it is seen that if P and Q divide AB internally and externally in the same ratio, then A and B divide PQ externally and internally in the same ratio; hence A and B are harmonic conjugates of P and Q.

In other words: *if* AB *is divided harmonically at* P *and* Q, *then* PQ *is divided harmonically at* A *and* B.

Example 1. *If* AB *is divided internally at* P *and externally at* Q *in the same ratio, then* AB *is the harmonic mean between* AQ *and* AP.



For, *by hypothesis*, $\qquad AQ : QB = AP : PB;$

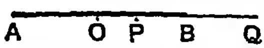
$\therefore$, *alternately*, $\qquad AQ : AP = QB : PB,$

that is, $\qquad AQ : AP = AQ - AB : AB - AP;$

$\therefore$ AQ, AB, AP are in Harmonic Progression.

Example 2. *If* AB *is divided harmonically at* P *and* Q, *and* O *is the middle point of* AB;

then $OP \cdot OQ = OA^2$.



For since AB is divided harmonically at P and Q,

$$\therefore AP : PB = AQ : QB:$$

$$\therefore AP - PB : AP + PB = AQ - QB : AQ + QB,$$

or, $\qquad 2OP : 2OA = 2OA : 2OQ;$

$$\therefore OP \cdot OQ = OA^2.$$

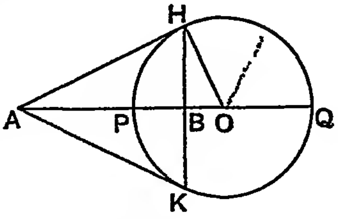
Conversely, if $\qquad OP \cdot OQ = OA^2,$

it may be shewn that

$$AP : PB = AQ : QB;$$

that is, that AB is divided harmonically at P and Q.

EXAMPLE 3. *The Arithmetic, Geometric and Harmonic means of two straight lines may be thus represented graphically.*



Let AP, AQ be the given lines, whose Arithmetic, Geometric, and Harmonic Means are to be found.

On PQ as diameter draw a circle; and from A draw the tangents AH, AK.

Draw the chord of contact HK, cutting AQ at B.

Join OH.

Then (i) AO is the Arithmetic mean between AP and AQ:
for clearly $AO = \frac{1}{2}(AP + AQ)$.

(ii) AH is the Geometric mean between AP and AQ:
for $AH^2 = AP \cdot AQ$. *Theor.* 58.

(iii) AB is the Harmonic mean between AP and AQ:
for, from the similar rt.-angled $\triangle^s$ AOH, HOB,

$$OA \cdot OB = OH^2 \qquad \textit{Theor. 66, Cor.}$$
$$= OP^2.$$

$\therefore$ PQ is cut harmonically at A and B; Ex. 2. p. 324.

$\therefore$ also AB is cut harmonically at P and Q.

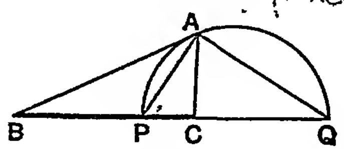
That is, AB is the Harmonic mean between AP and AQ.

And from the similar rt.-angled triangles OAH, HAB,

$$AO \cdot AB = AH^2; \qquad \textit{Theor. 66, Cor.}$$

$\therefore$ *the Geometric mean between two straight lines is the mean proportional between their Arithmetic and Harmonic means.*

EXAMPLE 4. *Given the base of a triangle and the ratio of the other sides, to find the locus of the vertex.*



Let BC be the given base, and let BAC be any triangle standing upon it, such that BA : AC = the given ratio.

It is required to find the locus of A.

Bisect the $\angle$BAC internally and externally by AP, AQ.

Then BC is divided internally at P, and externally at Q, so that BP : PC = BQ : QC = the given ratio;

$\therefore$ P and Q are fixed points.

And since AP, AQ bisect the $\angle$BAC internally and externally,

$\therefore$ the $\angle$PAQ is a rt. angle;

$\therefore$ the locus of A is the circle described on PQ as diameter.

EXERCISES ON HARMONIC SECTION.

1. If AB is divided harmonically at X and Y, shew that

$$\text{(i)}\quad \frac{2}{AB}=\frac{1}{AX}+\frac{1}{AY}$$

$$\text{(ii)}\quad \frac{2}{XY}=\frac{1}{BY}+\frac{1}{AY}.$$

2. X and Y are harmonic conjugates of A and B;

(i) if $AB=2{\cdot}4''$, and $AX=1{\cdot}5''$, find AY;

(ii) if $XY=1{\cdot}5$ cm., and $AY=2$ cm., find BY.

3. Any straight line is cut harmonically by the arms of an angle and its internal and external bisectors.

4. Given three points B, P, C in a straight line: find tne locus of points at which BP and PC subtend equal angles.

5. If through the middle point of the base of a triangle any line is drawn intersecting one side of the triangle, the other produced, and the line drawn parallel to the base from the vertex, it is divided harmonically.

6. If from either base angle of a triangle a line is drawn intersecting the median from the vertex, the opposite side, and the line drawn parallel to the base from the vertex, it is divided harmonically.

7. P, Q are harmonic conjugates of A and B, and C is an external point; if the angle PCQ is a right angle, shew that CP, CQ are the internal and external bisectors of the angle ACB.

8. AB is a given straight line, bisected at O, and divided harmonically at X and Y.

Trace the change of position of Y as X moves from O to B.

Taking AB equal to 20 cm. draw a graph to illustrate the variations of OY as OX changes.

9. Justify the following construction for finding the harmonic mean between two straight lines of given length.

Let AB and CD be the given lines, and let them be placed so as to be parallel. Join their ends towards the same parts by AC and BD, and towards opposite parts by AD and BC, cutting at O. Then if POQ is drawn parallel to the given lines and terminated by AC and BD, PQ is the required harmonic mean.

DEFINITIONS.

1. A series of points in a straight line is called a range. If the range consists of four points, of which one pair are harmonic conjugates with respect to the other pair, it is said to be a harmonic range.

2. A series of straight lines drawn through a point is called a pencil.

The point of concurrence is called the vertex of the pencil, and each of the straight lines is called a ray.

A pencil of four rays drawn from any point to a harmonic range is said to be a harmonic pencil.

3. A straight line drawn to cut a system of lines is called a transversal.

4. A system of four straight lines, no three of which are concurrent, is called a complete quadrilateral.

These straight lines will intersect two and two in *six* points, called the vertices of the quadrilateral; and each of the *three* straight lines which join the opposite vertices is called a diagonal.

THEOREMS ON HARMONIC SECTION.

1. *If a transversal is drawn parallel to one ray of a harmonic pencil, the other three rays intercept equal parts upon it: and conversely.*

2. *Any transversal is cut harmonically by the rays of a harmonic pencil.*

3. *In a harmonic pencil, if one ray bisect the angle between the other pair of rays, it is perpendicular to its conjugate ray. Conversely, if one pair of rays form a right angle, then they bisect internally and externally the angle between the other pair.*

4. *If* A, P, B, Q *and* a, p, b, q *are harmonic ranges, one on each of two given straight lines, and if* Aa, Pp, Bb, *the straight lines which join three pairs of corresponding points, meet at* S; *then will* Qq *also pass through* S.

5. *If two straight lines intersect at* A, *and if* A, P, B, Q *and* A, p, b, q *are two harmonic ranges one on each straight line (the points corresponding as indicated by the letters), then* Pp, Bb, Qq *will be concurrent: also* Pq, Bb, Qp *will be concurrent.*

6. *Use the last result to prove that in a complete quadrilateral each diagonal is cut harmonically by the other two.*

V. CENTRES OF SIMILITUDE.

EXAMPLE 1. *In two circles if any two parallel radii are drawn (one in each circle), the straight line joining their extremities cuts the line of centres in one or other of two fixed points.*

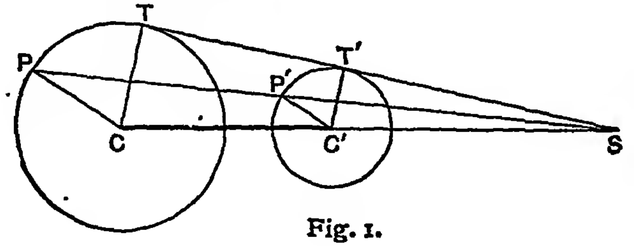


Fig. 1.

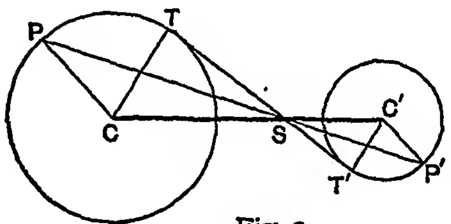


Fig. 2.

Take two circles whose centres are C and C′, and radii r and r' respectively; and let CP, C′P′ be any two par[l] radii drawn in the *same sense* in Fig. 1, and in *opposite senses* in Fig. 2. Let PP′ cut CC′ at S.

It is required to prove that (whatever be the direction of CP, CP′*)* S *is in one or other of two fixed positions.*

Proof. In both Figs. the $\triangle^s$ SCP, SC′P′ are equiangular;

$$\therefore\ SC : SC' = CP : C'P'$$
$$= r : r'.$$

Hence S divides CC′ $\begin{cases}\text{externally in Fig. 1}\\ \text{internally in Fig. 2}\end{cases}$ in the fixed ratio $r : r'$.

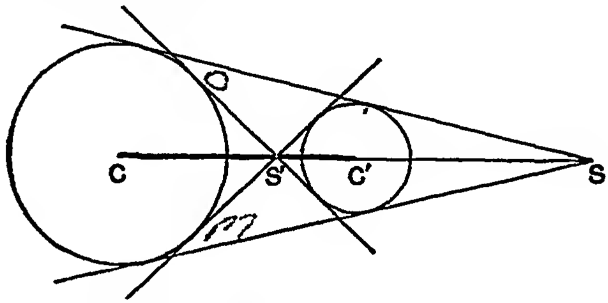
$\therefore$ in each Fig., S is a *fixed point* for all directions of CP, C′P′.

COROLLARY. Let TT′ be a common tangent to the two circles, *direct* in Fig. 1, and *transverse* in Fig. 2.

Then in both cases the radii CT, C′T′ are par[l];

∴ TT′ cuts the line of centres at S.

DEFINITION. In the figure given below the points S and S′ which divide the line of centres of two circles externally and internally in the ratio of their radii are called Centres of Similitude, the former being the centre of *direct* and the latter of *transverse* similitude.



COROLLARY. Since $\dfrac{SC}{SC'}=\dfrac{r}{r'}=\dfrac{S'C}{S'C'}$,

the centres of the circles and the centres of similitude form an harmonic range.

Hence the transverse and direct common tangents intersect on the line of centres at points which divide that line harmonically.

EXERCISES.

1. From centres C and C′, 5·5 cm. apart, draw two circles of radii 3·2 cm. and 1·2 cm. respectively, and determine (i) graphically (ii) by calculation the distances of their centres of similitude from C.

2. Two circles whose centres are C and C′ respectively have radii 1·8″ and 1·0″, and their direct centre of similitude is 2·7″ distant from C. Find the distance (i) between their centres (ii) between their centres of similitude.

3. In Fig. 1 of the preceding page, if SP cuts the circles (C) and (C′) again at Q and Q′ respectively, shew that

$$SQ.SP'=SP.SQ'=ST.ST'.$$

EXERCISES.

(On Centres of Similitude. Continued.)

4. In the triangle ABC, I is the in-centre, and I_1 the ex-centre opposite to A. If AI_1 cuts BC at Y, shew that A and Y are the centres of similitude of the two circles.

5. *Shew that the orthocentre and centroid of a triangle are respectively the external and internal centres of similitude of the circumscribed and nine-points circle.*

6. *If a variable circle touches two fixed circles, the line joining the points of contact passes through a centre of similitude. Distinguish between the different cases.*

7. *Describe a circle which shall touch two given circles and pass through a given point.*

8. *Describe a circle which shall touch three given circles.*

9. C_1, C_2, C_3 *are the centres of three given circles;* S'_1, S_1, *are the internal and external centres of similitude of the pair of circles whose centres are* C_2, C_3, *and* S'_2, S_2, S'_3, S_3, *have similar meanings with regard to the other two pairs of circles: shew that*

(i) S'_1C_1, S'_2C_2, S'_3C_3 *are concurrent;*

(ii) *the six points* S_1, S_2, S_3, S'_1, S'_2, S'_3, *lie three and three on four straight lines.* [See Theorems IX. and X., pp. 344, 345.]

ORTHOGONAL CIRCLES.

DEFINITION. Circles which intersect at a point, so that the two tangents at that point are at right angles to one another, are said to be **orthogonal**, or to **cut one another orthogonally**.

1. If two circles cut one another orthogonally, the tangent to each circle at a point of intersection will pass through the centre of the other circle.

2. If two circles cut one another orthogonally, the square on the distance between their centres is equal to the sum of the squares on their radii.

3. Find the locus of the centres of all circles which cut a given circle orthogonally at a given point.

4. Describe a circle to pass through a given point and cut a given circle orthogonally at a given point.

VI. POLE AND POLAR.

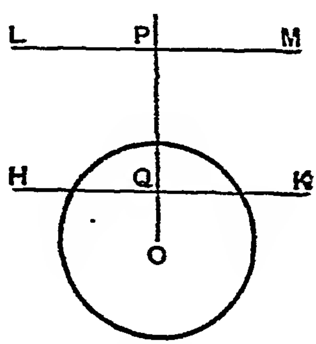
DEFINITIONS.

1. If in any straight line drawn from the centre of a circle two points are taken such that the rectangle contained by their distances from the centre is equal to the square on the radius, each point is said to be the inverse of the other.

Thus in the figure given below, if O is the centre of the circle, and if $OP \cdot OQ = (\text{radius})^2$, then each of the points P and Q is the inverse of the other.

It is clear that if one of these points is within the circle the other must be without it.

2. The polar of a given point with respect to a given circle is the straight line drawn through the inverse of the given point at right angles to the line which joins the given point to the centre: and with reference to the polar the given point is called the pole.



Thus in the adjoining figure, if $OP \cdot OQ = (\text{radius})^2$, and if through P and Q, LM and HK are drawn perp. to OP; then HK is the polar of the point P, and P is the pole of the st. line HK with respect to the given circle; also LM is the polar of the point Q, and Q the pole of LM.

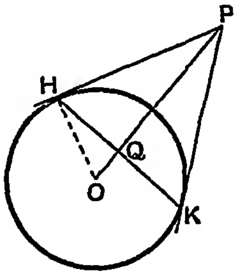
It is clear that the polar of an *external* point must intersect the circle, and that the polar of an *internal* point must fall without it: also that the polar of a point *on the circumference* is the tangent at that point.

EXAMPLE 1. *The polar of an external point with reference to a circle is the chord of contact of tangents drawn from the given point to the circle.*

From the external point P let two tangents PH, PK be drawn to a circle of which O is the centre.

Join HK.

It is required to prove that HK *is the polar of* P.



Now OP evidently cuts the chord of contact HK at right angles at Q.

Join OH.

Then from the similar rt.-angled $\triangle^s$ POH, HOQ,

$$OP : OH = OH : OQ;$$

$$\therefore\ OP \cdot OQ = (\text{radius})^2;$$

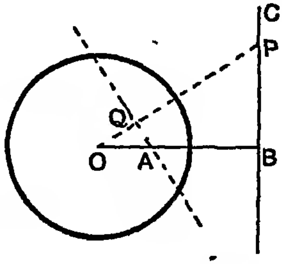
hence HK is the polar of P.

Q.E.D.

EXAMPLE 2. *If* A *and* P *are any two points, and if the polar of* A *with respect to any circle passes through* P, *then the polar of* P *must pass through* A.

Let BC be the polar of the point A with respect to a circle whose centre is O, and let BC pass through P.

It is required to prove that the polar of P *passes through* A.



Join OP; and from A draw AQ perp. to OP. We shall shew that AQ is the polar of P.

Now since BC is the polar of A,

$\therefore$ the $\angle$ABP is a rt. anglo; *Def.* 2, page 331,

and the $\angle$AQP is a rt. angle: *Constr.*

$\therefore$ the four points A, B, P, Q are concyclic;

$$\therefore\ OQ \cdot OP = OA \cdot OB \qquad \textit{Theor. } 58.$$

$$= (\text{radius})^2, \text{ for CB is the polar of A.}$$

And since AQ is perp. to OP,

$\therefore$ AQ is the polar of P.

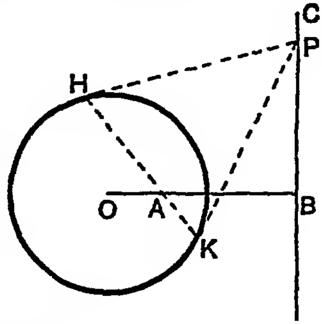
That is, the polar of P passes through A.

Q.E.D.

NOTE. A similar proof applies to tho case when the given point A is without the circle, and the polar BC cuts it.

The above Theorem is known as the Reciprocal Property of Pole and Polar.

EXAMPLE 3. *The locus of the intersection of tangents drawn to a circle at the extremities of all chords which pass through a given point within the circle is the polar of that point.*



Let A be the given point within the circle. Let HK be *any* chord passing through A; and let the tangents at H and K intersect at P.

It is required to prove that the locus of P *is the polar of the point* A.

(α) To shew that P lies on the polar of A.

Since HK is the chord of contact of tangents drawn from P,

$\therefore$ HK is the polar of P. Ex. 1, p 332.

But HK, the polar of P, passes through A;

$\therefore$ the polar of A passes through P: Ex. 2, p. 332.

that is, the point P lies on the polar of A.

(β) To shew that *any* point on the polar of A satisfies the given conditions.

Let BC be the polar of A, and let P be any point on it.

Draw tangents PH, PK, and let HK be the chord of contact.

Now from Ex. 1, p. 332, we know that the chord of contact HK is the polar of P,

and we also know that the polar of P must pass through A; for P is on BC, the polar of A: Ex. 2, p. 332.

that is, HK passes through A.

$\therefore$ P is the point of intersection of tangents drawn at the extremities of a chord passing through A.

From (α) and (β) we see that the required locus is the polar of A.

NOTE. If A is outside the circle the theorem (α) still holds good; but the converse theorem (β) is not true for all points in BC. For if A is without the circle, the polar BC will intersect it; and no point on that part of the polar which is within the circle can be a point of intersection of tangents.

We now see that

(i) *The Polar of an* external *point with respect to a circle is the chord of contact of tangents drawn from it.*

(ii) *The Polar of an* internal *point is the locus of the intersections of tangents drawn at the extremities of all chords which pass through it.*

(iii) *The Polar of a* point on the circumference *is the tangent at that point.*

EXAMPLE 4. *Any chord of a circle through a fixed point* P *is divided harmonically by* P *and the polar of* P.

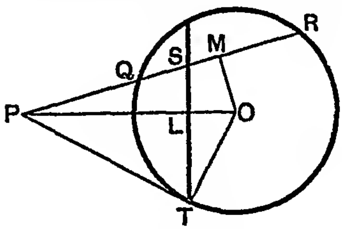


Fig. 1.

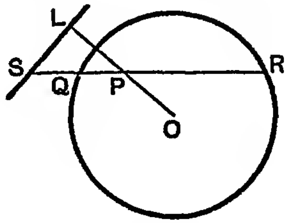


Fig. 2.

Let O be the centre of the given circle, and let QR be a chord passing through the given point P.

(i) When P is an external point (Fig. 1).

Draw a tangent PT, and let the polar of P cut PO in L, and QR in S.

It is required to prove that QR *is divided harmonically at* P *and* S

Draw OM perp. to QR, and join OT.

Then $PQ \,.\, PR = PT^2$

$= PL \,.\, PO$, since PTO is a rt. $\angle$,

$= PM \,.\, PS$, since S, L, O, M are concyclic.

$\therefore\ 2PQ \,.\, PR = 2PM \,.\, PS$

$= (PQ + PR)\, PS$; Ex. 9, p. 65.

$$\therefore\ PS = \frac{2PQ \,.\, PR}{PQ + PR};$$

$\therefore$ PQ, PS, PR are in Harmonical Progression;

that is, PS is divided harmonically at Q and R,

$\therefore$ also QR is divided harmonically at P and S.

(ii) When P is an internal point (Fig. 2).

Let SL be the polar of P.

Then since the polar of P passes through S, the polar of S passes through P.

$\therefore$ by the former case QR is divided harmonically at S and P.

The above theorem is known as the Harmonic Property of Pole and Polar.

DEFINITION.

A triangle so related to a circle that each side is the polar of the opposite vertex is said to be self-conjugate with respect to the circle.

EXERCISES ON POLE AND POLAR.

1. *The straight line which joins any two points is the polar with respect to a given circle of the point of intersection of their polars.*

2. *The point of intersection of any two straight lines is the pole of the straight line which joins their poles.*

3. *Find the locus of the poles of all straight lines which pass through a given point.*

4. *Find the locus of the poles, with respect to a given circle, of tangents drawn to a concentric circle.*

5. *If two circles cut one another orthogonally and* PQ *be any diameter of one of them; shew that the polar of* P *with regard to the other circle passes through* Q.

6. *If two circles cut one another orthogonally, the centre of each circle is the pole of their common chord with respect to the other circle.*

7. *Any two points subtend at the centre of a circle an angle equal to one of the angles formed by the polars of the given points.*

8. O *is the centre of a given circle, and* AB *a fixed straight line.*
P *is* any *point in* AB; *find the locus of the point inverse to* P *with respect to the circle.*

9. *Given a circle, and a fixed point* O *on its circumference:* P *is* any *point on the circle: find the locus of the point inverse to* P *with respect to any circle whose centre is* O.

10. *Given two points* A *and* B, *and a circle whose centre is* O; *shew that the rectangle contained by* OA *and the perpendicular from* B *on the polar of* A *is equal to the rectangle contained by* OB *and the perpendicular from* A *on the polar of* B.

11. *Four points* A, B, C, D *are taken in order on the circumference of a circle;* DA, CB *intersect at* P, AC, BD *at* Q, *and* BA, CD *in* R: *shew that the triangle* PQR *is self-conjugate with respect to the circle.*

12. *Give a* linear *construction for finding the polar of a given point with respect to a given circle. Hence find a linear construction for drawing a tangent to a circle from an external point.*

13. *If a triangle is self-conjugate with respect to a circle, the centre of the circle is at the orthocentre of the triangle.*

14. *The polars, with respect to a given circle, of the four points of a harmonic range form a harmonic pencil: and conversely.*

VII. THE RADICAL AXIS.

EXAMPLE 1. *To find the locus of points from which the tangents drawn to two given circles are equal.*

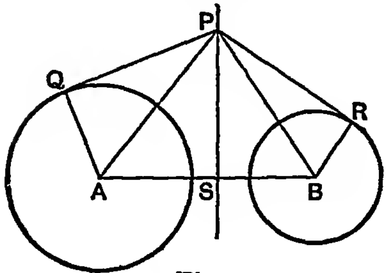


Fig. 1.

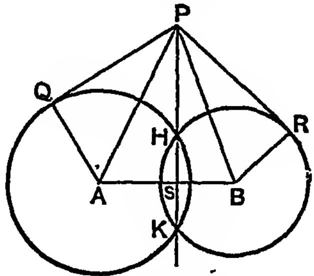


Fig. 2.

Let A and B be the centres of the given circles, whose radii are a and b; and let P be any point such that the tangent PQ drawn to the circle (A) is equal to the tangent PR drawn to the circle (B).

It is required to find the locus of P.

Join PA, PB, AQ, BR, AB;
from P draw PS perp. to AB.

Then because $PQ=PR$, $\therefore$ $PQ^2=PR^2$.

But $PQ^2=PA^2-AQ^2$; and $PR^2=PB^2-BR^2$: *Theor.* 29.

$\therefore$ $PA^2-AQ^2=PB^2-BR^2$;

that is, $PS^2+AS^2-a^2=PS^2+SB^2-b^2$; *Theor.* 29.

or, $AS^2-a^2=SB^2-b^2$.

Hence AB is divided at S, so that $AS^2-SB^2=a^2-b^2$:

$\therefore$ S is a *fixed point.*

Hence all points from which equal tangents can be drawn to the two circles lie on the straight line which cuts AB at rt. angles, so that the difference of the squares on the segments of AB is equal to the difference of the squares on the radii.

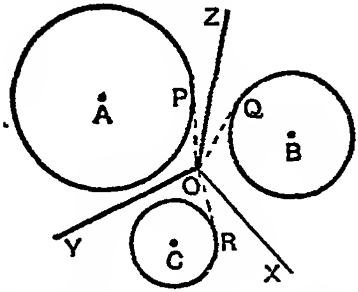
Again, by simply retracing these steps, it may be shewn that in Fig. 1 every point in SP, and in Fig. 2 every point in SP exterior to the circles, is such that tangents drawn from it to the two circles are equal.

Hence we conclude that in Fig. 1 the whole line SP is the required locus, and in Fig. 2 that part of SP which is without the circles.

In either case SP is said to be the **Radical Axis** of the two circles.

COROLLARY. *If the circles cut one another as in Fig. 2, it is clear that the Radical Axis is identical with the straight line which passes through the points of intersection of the circles;* for it follows readily from Theorem 58 that tangents drawn to two intersecting circles from any point in the common chord produced are equal.

EXAMPLE 2. *The Radical Axes of three circles taken in pairs are concurrent.*



Let there be three circles whose centres are A, B, C.

Let OZ be the radical axis of the ⊙ˢ (A) and (B);

and OY the Radical Axis of the ⊙ˢ (A) and (C), O being the point of their intersection.

It is required to prove that the radical axis of the ⊙ˢ (B) *and* (C) *passes through* O.

It will be found that the point O is either *without* or *within all the circles.*

I. When O is without the circles.

From O draw OP, OQ, OR tangents to the ⊙ˢ (A), (B), (C).

Then because O is a point on the radical axis of (A) and (B);

$\therefore$ OP=OQ.

And because O is a point on the radical axis of (A) and (C),

$\therefore$ OP=OR;

$\therefore$ OQ=OR;

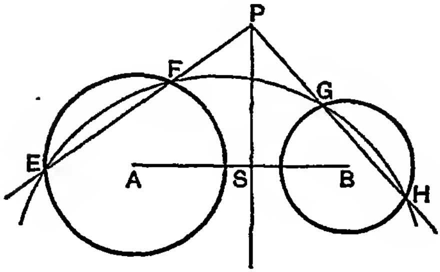
$\therefore$ O is a point on the radical axis of (B) and (C);

that is, the radical axis of (B) and (C) passes through O.

II. If the circles intersect in such a way that O is within them all; the radical axes are then the common chords of the three circles taken two and two; and it is required to prove that these common chords are concurrent. This may be shewn indirectly by Theorem 57.

DEFINITION. The point of intersection of the radical axes of three circles taken in pairs is called the radical centre.

EXAMPLE 3. *To draw the radical axis of two given circles.*



Let A and B be the centres of the given circles.

It is required to draw their radical axis.

(α) If the given circles intersect, then the st. line drawn through their points of intersection will be the radical axis.

(β) But if the given circles do not intersect,

describe any circle so as to cut them in E, F and G, H.

Join EF and HG, and produce them to meet in P.

Join AB; and from P draw PS perp. to AB.

Then PS *is the radical axis of the* ⊙ˢ (A), (B).

Proof. From the ⊙ EFGH, PE . PF = PH . PG.

Now the sq. on the tangent from P to the ⊙ (A) = PE . PF;

and the sq. on the tangent from P to the ⊙ (B) = PH . PG.

Hence the tangents from P to the ⊙ˢ (A) and (B) are equal;

∴ P is a point on the radical axis.

And since PS is perp. to the line of centres,

∴ PS is the radical axis. *Ex.* 1, p 336.

DEFINITION. If each pair of circles in a given system have the same radical axis, the circles are said to be co-axal.

EXERCISES ON THE RADICAL AXIS.

1. *Shew that the radical axis of two circles bisects any one of their common tangents.*

2. *If tangents are drawn to two circles from any point on their radical axis; shew that a circle described with this point as centre and any one of the tangents as radius, cuts both the given circles orthogonally.* [See Def. p. 330.]

3. O *is the radical centre of three circles, and from* O *a tangent* OT *is drawn to any one of them: shew that a circle whose centre is* O *and radius* OT *cuts all the given circles orthogonally.*

4. *If three circles touch one another, taken two and two, shew that their common tangents at the points of contact are concurrent.*

5. *If circles are described on the three sides of a triangle as diameter, their radical centre is the orthocentre of the triangle.*

6. *All circles which pass through a fixed point and cut a given circle orthogonally, pass through a second fixed point.*

7. *Find the locus of the centres of all circles which pass through a given point and cut a given circle orthogonally.*

8. *Describe a circle to pass through two given points and cut a given circle orthogonally.*

9. *Find the locus of the centres of all circles which cut two given circles orthogonally.*

10. *Describe a circle to pass through a given point and cut two given circles orthogonally.*

11. *The difference of the squares on the tangents drawn from any point to two circles is equal to twice the rectangle contained by the straight line joining their centres and the perpendicular from the given point on their radical axis.*

12. *In a system of co-axal circles which do not intersect, any point is taken on the radical axis; shew that a circle described from this point as centre, with radius equal to the tangent drawn from it to any one of the circles, will meet the line of centres in two* fixed *points.*

[*These fixed points are called the* Limiting Points *of the system.*]

13. *In a system of co-axal circles the two limiting points and the points in which any one circle of the system cuts the line of centres form a harmonic range.*

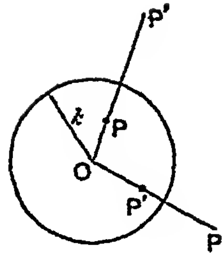
14. *In a system of co-axal circles a limiting point has the same polar with regard to all the circles of the system.*

15. *If two circles are orthogonal any diameter of one is cut harmonically by the other.*

VIII. INVERSION.

DEFINITIONS.

1. If from any fixed point O a straight line OP is drawn, and a point P′ is taken on OP, or OP produced, such that OP . OP′ = k^2, where k is constant, then each of the points P and P′ is said to be the inverse of the other with respect to the circle whose centre is O and radius k.



2. The point O is called the origin of inversion, and k is called the radius of inversion. Also k^2 is sometimes referred to as the constant of inversion.

3. If P traces out a locus, to every position of P there is a corresponding position of P′. The locus of P′ is called the inverse of the locus of P.

From Definition 1 it is clear that any straight line passing through the origin is its own inverse.

EXAMPLE 1. *To find the inverse of a straight line not passing through the origin of inversion.*

Let P be any point on the given st. line AB, O the origin, and k the radius of inversion.

Draw OQ perp. to the given line. Take P′ and Q′ the inverses of P and Q.

Join P′Q′.

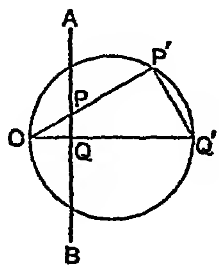
Then OP . OP′ = k^2

= OQ . OQ′.

∴ the pts. P, P′, Q, Q′ are concyclic;

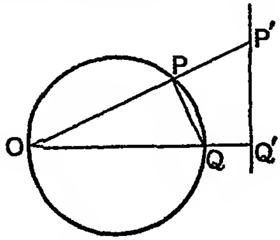
∴ the ∠OP′Q′ = the ∠OQP

= a rt. ∠.



Hence the locus of P′ is a circle which passes through O, such that the diameter OQ′ is perp. to the given line.

Example 2. *To find the inverse of a circle with respect to a point on its circumference.*



Let OQ be the diameter of the given circle which passes through the origin O.

Take any point P on this circle; and with k as radius of inversion, let Q' and P' be the inverses of Q and P.

Join PQ, P'Q'.

Then $\quad OP \cdot OP' = k^2$

$= OQ \cdot OQ'$.

$\therefore$ the pts. P, P', Q, Q' are concyclic;

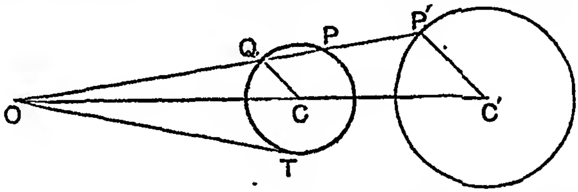
$\therefore$ the $\angle OQ'P' =$ the $\angle OPQ$

$=$ a rt. $\angle$.

$\therefore$ P'Q' is perp. to OQ'.

Hence the locus of P' is a st. line perp. to the diameter through the origin.

Example 3. *To find the inverse of a circle with respect to a point not on the circumference.*



Let O be the origin and P any point on the given circle whose centre is C.

Let P' be the inverse of P, so that $OP \cdot OP' = k^2$.

Let OP meet the given circle again in Q. Join QC.

Draw OT a tangent to the circle, and let $OT = t$.

Then $\quad OP \cdot OP' = k^2$, and $OP \cdot OQ = t^2$.

$$\therefore \frac{OP \cdot OP'}{OP \cdot OQ} = \frac{k^2}{t^2};$$

$\therefore OP' : OQ = k^2 : t^2$.

Draw P'C' par[l] to QC to meet OC produced in C'.

Then $\quad OC' : OC = OP' : OQ$

$= k^2 : t^2$.

$\therefore$ C' is a fixed point.

Also $\quad C'P' : CQ = OP' : OQ$

$= k^2 : t^2$.

$\therefore$ C'P' is constant, and the locus of P' is a circle whose centre is C'.

Corollary. The origin is a centre of similitude of the circle and its inverse.

EXAMPLE 4. *Any line through the origin of inversion cuts two inverse loci at the same angle on opposite sides of the line.*

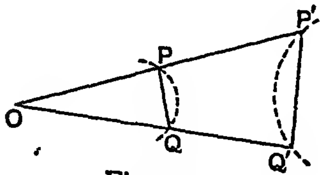


Fig. 1.

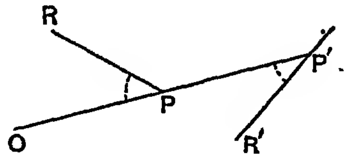


Fig. 2.

Let P and Q be two points on a locus, and let P′, Q′ be their inverses with respect to O.

Then $$\text{OP} \cdot \text{OP}' = k^2$$
$$= \text{OQ} \cdot \text{OQ}'.$$

$\therefore$ the pts. P, P′, Q, Q′ are concyclic;

$\therefore$ $\angle \text{OQP} =$ the $\angle \text{OP}'\text{Q}'$.

Now let Q move up to P, so that the st. line QP ultimately becomes the tangent at P to the locus of P. Then at the same time the st. line Q′P′ becomes the tangent at P′ to the locus of P′.

Hence in Fig. 2, if PR and P′R′ are the tangents at P and P′,

the $\angle \text{OPR} =$ the $\angle \text{OP}'\text{R}'$.

that is, OPP′ cuts the loci of P and P′ at the same angle *on opposite sides of* OPP′.

COROLLARY. *At any point of intersection two curves cut at the same angle as their inverses at the inverse point. Also if two curves touch at* P *their inverses touch at the inverse point* P′.

EXAMPLE 5. *To express the distance between two points in terms of the distance between their inverses and the distances of these points from the origin.*

If P′, Q′ are the inverses of P, Q, [Fig. 1 of Example 4.]

$$\text{OP} \cdot \text{OP}' = k^2 = \text{OQ} \cdot \text{OQ}';$$

and from the similar triangles OPQ, OQ′P′,

$$\frac{\text{P}'\text{Q}'}{\text{PQ}} = \frac{\text{OP}'}{\text{OQ}} = \frac{\text{OP} \cdot \text{OP}'}{\text{OP} \cdot \text{OQ}} = \frac{k^2}{\text{OP} \cdot \text{OQ}}.$$

$$\therefore \text{P}'\text{Q}' = \frac{k^2 \cdot \text{PQ}}{\text{OP} \cdot \text{OQ}}.$$

EXERCISES ON INVERSION.

1. If O, P, Q, R are collinear points and P′, Q′, R′ the inverses of P, Q, R with respect to O, prove that

(i) if OP, OQ, OR are in Arithmetical Progression then OP′, OQ′, OR′ are in Harmonical Progression.

(ii) if OP, OQ, OR are in Geometrical Progression, then OP′, OQ′, OR′ are also in Geometrical Progression.

2. Find the inverse of the circum-circle of an isosceles triangle with respect to the vertex of the triangle as origin.

3. Shew that a circle can be inverted into itself with respect to any point O as origin.

[Take k equal to the length of the tangent from O.]

4. Shew that a circle inverts into itself with respect to the centre of any orthogonal circle.

5. AB is a chord of a circle bisected at O. Shew that, with O as origin, and OA as radius of inversion, the circle inverts into itself.

6. Shew that any two circles can be inverted into themselves.

[See Ex. 1. p. 336. Take the origin O on SP, and take k equal to the length of the tangent from O.]

7. Shew that any three circles can be inverted into themselves.

[See Ex. 2. p. 337.]

8. Shew that if a straight line cuts a circle, each may be inverted into the other by suitable selection of the origin and constant of inversion.

9. Shew that any three circles may be inverted into three circles whose centres are collinear.

[See Ex. 3, p. 339, and take the origin of inversion on the orthogonal circle.]

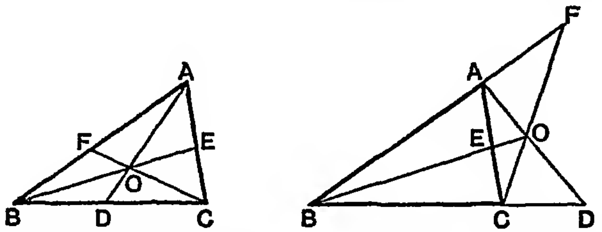
10. Shew that the diameters of a circle may be inverted into a series of co-axal circles orthogonal to the inverse of the given circle.

11. P, Q, R are three points taken in order on a straight line. Find the inverse of the statement

$$PQ+QR=PR.$$

√IX. CEVA'S THEOREM.

If three concurrent straight lines are drawn from the angular points of a triangle to meet the opposite sides, then the product of three alternate segments taken in order is equal to the product of the other three segments.



Let AD, BE, CF be drawn from the vertices of the $\triangle ABC$ to intersect at O, and cut the opposite sides at D, E, F.

It is required to prove that

$$BD \cdot CE \cdot AF = DC \cdot EA \cdot FB.$$

Now the $\triangle^s$ AOB, AOC have a common base AO; and it may be shewn, by drawing perpendiculars from B and C to AD, that

$$BD : DC = \text{the alt. of } \triangle AOB : \text{the alt. of } \triangle AOC;$$

$$\therefore \frac{BD}{DC} = \frac{\triangle AOB}{\triangle AOC};$$

similarly,
$$\frac{CE}{EA} = \frac{\triangle BOC}{\triangle BOA};$$

and
$$\frac{AF}{FB} = \frac{\triangle COA}{\triangle COB}.$$

Multiplying these ratios, we have

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1;$$

or,
$$BD \cdot CE \cdot AF = DC \cdot EA \cdot FB.$$

NOTE. The converse of this theorem, which may be proved indirectly, is very important; it may be enunciated thus:

If three straight lines drawn from the vertices of a triangle cut the opposite sides so that the product of three alternate segments taken in order is equal to the product of the other three, then the three straight lines are concurrent.

That is, if $BD \cdot CE \cdot AF = DC \cdot EA \cdot FB$,

then AD, BE, CF *are concurrent.*

√X. MENELAUS' THEOREM.

If a transversal is drawn to cut the sides, or the sides produced, of a triangle, the product of three alternate segments taken in order is equal to the product of the other three segments.

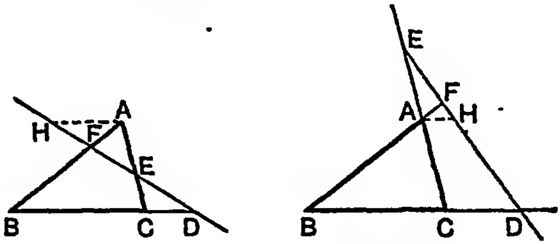


Fig. 1. Fig. 2.

Let ABC be a triangle, and let a transversal meet the sides BC, CA, AB, or these sides produced, at D, E, F.

It is required to prove that

$$BD \cdot CE \cdot AF = DC \cdot EA \cdot FB.$$

Draw AH par[l] to BC, meeting the transversal at H.

Then from the similar $\triangle$s DFB, HAF,

$$\frac{AF}{FB} = \frac{AH}{BD};$$

and from the similar $\triangle$s DCE, HAE,

$$\frac{CE}{EA} = \frac{CD}{AH}.$$

$\therefore$, by multiplication, $\quad \frac{CE}{EA} \cdot \frac{AF}{FB} = \frac{CD}{BD};$

that is, $\quad \frac{BD \cdot CE \cdot AF}{DC \cdot EA \cdot FB} = 1,$

or, $\quad BD \cdot CE \cdot AF = DC \cdot EA \cdot FB.$

NOTE. In this theorem the transversal must either meet two sides and the third side produced, as in Fig. 1; or all three sides produced, as in Fig. 2.

The converse of this theorem may be proved indirectly:

If three points are taken in two sides of a triangle and the third side produced, or in all three sides produced, so that the product of three alternate segments taken in order is equal to the product of the other three segments, the three points are collinear.

DEFINITIONS.

1. If two triangles are such that the three straight lines joining corresponding vertices are concurrent, they are said to be **co-polar.**

2. If two triangles are such that the three points of intersection of corresponding sides are collinear, they are said to be **co-axial.**

EXERCISES.

1. By means of Ceva's Theorem prove the following properties of a triangle.

   (i) The perpendiculars to the sides from their middle points are concurrent.

   (ii) The bisectors of the angles are concurrent.

   (iii) The medians are concurrent

2. D, E, F are the points of contact of the in-circle of a triangle with the sides BC, CA, AB respectively. If EF, FD, DE meet these sides respectively in P, Q, R, shew that P, Q, R are collinear.

3. With the same notation as in Example 2, shew that the points B, D, C, P form a harmonic range.

4. If the tangents at A, B, C of the circum-circle of the triangle ABC meet the opposite sides in D, E, F respectively, shew that

$$BD : CD = BA^2 : AC^2.$$

Hence prove that D, E, F are collinear.

5. *The straight lines which join the vertices of a triangle to the points of contact of the inscribed circle (or any of the three escribed circles) are concurrent.*

6. *The middle points of the diagonals of a complete quadrilateral are collinear.* [See Def. 4, p. 327.]

7. *Shew that each diagonal of a complete quadrilateral is divided harmonically by the other two diagonals.*

8. *Co-polar triangles are also co-axial; and conversely co-axial triangles are also co-polar.*

9. *The six centres of similitude of three circles lie three by three on four straight lines.*

ANSWERS TO NUMERICAL EXERCISES.

Since the utmost care cannot ensure absolute accuracy in graphical work, results so obtained are likely to be only approximate. The answers here given are those found by calculation, and being true so far as they go, furnish a standard by which the student may test the correctness of his drawing and measurement. Results within one per cent. of those given in the Answers may usually be considered satisfactory.

Exercises. Page 145.

1. 5 cm. 2. 24″. 3. 0·6″, 0·8″. 4. $\sqrt{7}=2{\cdot}6$ cm.
5. 1 ft. 6. 0·6 sq. in. 7. 0·8″.

Exercises. Page 149.

1. 1·7″. 2. $3\sqrt{2}=4{\cdot}2$ cm. 3. $2\sqrt{3}=3{\cdot}5$ cm.
4. 17″. 6. 5 cm.

Exercises. Page 151.

6. 4 cm. 7. 1·3″.

Exercises. Page 153.

2. 1·85″. 3. 1·62″. 5. 0·85″; (2·1″, 2·1″); 2·97″.

Exercises. Page 155.

5. 51″ 6. 1·6″; 1·5″, 0·6″.

Exercises. Page 157.

4. (8, 11). 5. 17; 10; (0, −8).

Exercises. Page 161.

1. 74°, 148°, 16°. 2. 115°, 230°. 3. 55°, 8°, 47°.

Exercises. Page 177.

1. 8·0 cm. 2. 0·6″. 3. 8·7 cm. 4. 12″, 67°. 5. 2·5″.

Exercises. Page 179.

3. 3 cm. and 17 cm.

Exercises. Page 181.

1. 72°, 108°, 108°.

Exercises. Page 187.

2. 1·6″. 3. 1·7″. 4. 1·98″, 1·6″.

Exercises. Page 198.

2. 2·3 cm., 4·6 cm., 6·9 cm. 3. 1·39″.
4. 6·9 cm.; 20·78 sq. cm. 7. 3·2 cm.

Exercises. Page 199.

1. 2·12″; 4·50 sq. in. 4. 8·5 cm. 5. 2·0″.

Exercises. Page 200.

4. $128\frac{4}{7}$°; 1·73″.

Exercises. Page 201.

1. 3·46″; 4·00″. 2. 259·8 sq. cm.
4. (i) 41·57 sq. cm.; (ii) 77·25 sq. cm.

Exercises. Page 205.

1. (i) 28·3 cm.; (ii) 628·3 cm. 2. (i) 16·62 sq. in.; (ii) 352·99 sq. in.
3. 11·31 cm.; 10·18 sq. cm. 4. 56 sq. cm. 5. 43·98 sq. in.
7. 30·5 sq. cm. 8. 8·9″. 9. 4″; 3″. 10. 12·57 sq. in.
11. Circumferences, 4·4″, 6·3″. Areas, 1·54 sq. in., 3·14 sq. in.

Exercises. Page 225.

3. 6·4 sq. cm. 4. 3·7″. 5. 10 cm. 6. 1″.

Exercises. Page 228.

1. 630 sq. cm. 15 cm.

Exercises. Page 231.

2. 8·5 cm. 90°. 3. A circle of rad. 6 cm.
4. 5·20″. 6. 0·25″.

Exercises. Page 235.

1. (i) 16 sq. cm. (ii) 16 sq. cm. 2. (i) 16 sq. cm. (ii) 16 sq. cm.
3. 0·8″. 4. (i) 1·2″. (ii) 12·5 cm. 5. (i) 1·6″, 4·1″. (ii) 3·5 cm.
6. Two concentric circles, radii 2 cm. and 6 cm.

Exercises. Page 237.

1. 26″. 2. 48 ft.; 8 ft. 3. 2 cm.; 32 cm.
4. 3·6″. 5. 8100 miles; 10 miles.

Exercises. Page 239.

1. 4 cm. 2. 2·12″. 3. 1·94″. 4. 1·97″.
5. 6·6 cm. 6. 6·6, 2·4. 7. 35·2, 4·8. 8. 3·5 cm.
9. 11·2, 3·2. 10. 9·6, 2·6.

Exercises. Page 241.

1. 2·47″. 2. −3·24″.

Exercises. Page 245.

1. 8, 2. 2. 7, 7. 3. 9·3, 2·7.
4. 9, −4. 5. 11·32, −4·32. 6. 7·24, 2·76.

Exercises. Page 246.

1. 6. 2. 36, 45. 3. 16, 12.
4. (10, $12\frac{1}{2}$); $12\frac{1}{2}$. 5. (17, 18); $12\sqrt{2}=16·97$.
7. 15. 8. 10. 9. Four. (26, 15). 10. 12·84.

Exercises. Page 253.

1. (i) 35; (ii) 8; (iii) a.
3. 4·0″, 5·6″.
4. 16·5 cm., 12·0 cm.
5. 4·0 cm., 2·4 cm.; 16·0 cm., 9·6 cm.

Exercises. Page 258.

1. (i) each $=3:2$; (ii) each $=5:3$; (iii) each $=5:2$.
2. (i) 1·4″; (ii) 0·8″; (iii) 6·4 cm., 2·4 cm.
3. (i) 5·6 cm. (ii) 7·7 cm, 2 8 cm.

Exercises. Page 259.

1. 0·9″, 0·6″; 4·5″, 3·0″; 3 : 2.
2. 2·0 cm., 1·5 cm.; 14·0 cm., 10·5 cm.

Exercises. Page 262.

1. (i) 1·2″; (ii) 2·0″; (iii) 7·7 cm.
2. (i) 2·1″; (ii) 6·3 cm.
3. QB=3·5″, BR=2·5″.
4. 3·2 cm., 4·2 cm.
5. 2·1″, 1·8″.
6. 5 ft., $12\frac{3}{4}$ ft., $9\frac{1}{4}$ ft.
7. 1·2″, 1·3″, 1·95″.
8. $5\frac{5}{8}$ cm.
9. 0·8 cm., 1·4 cm., 2·1 cm.

Exercises. Page 271.

1. 17, $\frac{8}{17}$, $\frac{15}{17}$, $\frac{8}{15}$.
2. 37. sinc $=\frac{12}{37}$, cosine $=\frac{35}{37}$, tangent $=\frac{12}{35}$.
4. $\frac{12}{13}$, $\frac{5}{12}$, $\frac{77}{85}$, $\frac{77}{36}$.
6. 37°.
7. 35°, 26°, 45°.
8. A=58°, cos A=0·53.
10. AC=7·8, A=39°, sin A=0·63, cos A=0·78.
11. 3·4, 35°.

Exercises. Page 278.

1. (i) 1·0″; (ii) 0·9″; (iii) 6·0 cm
2. 1·4″, 0·6″; 3·5″, 1·5″.
3. (i) 2·0; (ii) 2·8; (iii) 20.
4. 1·6 cm., 2·4 cm., 3·2 cm.
5. 1·8″, 1·2″, 0·9″.
6. 2·7″
7. (i) 1·73; (ii) 3·16; (iii) 1·67.
8. (i) 3; (ii) 3·21; (iii) 2·26.
9. (i) 1·2″, 1·6″, 2·0″; (ii) 3·0 cm., 3·6 cm., 4·5 cm.
   (iii) 2·5 cm., 4·3 cm., 5·0 cm. (iv) $b=3·4″$, $c=2·1″$, nearly.

Exercises. Page 279.

1. 140 m., 160 m.; 125 m.
2. $12\frac{1}{2}$ yds.
3. $42\frac{1}{2}$ miles
4. 30 ft., 4 ft.
5. 24 ft., 2 ft. 4 in.
6. 60 ft.
7. 72 ft.
8. 106 ft.

Exercises. Page 284.

3. 0·52. 5. 31 : 28, nearly.

Exercises. Page 287.

1. 10·5 sq. in. 2. 3·0 cm. 3. 64 sq. cm.
4. 11·0″. 5. 33·9 acres.

Exercises. Page 289.

6. 4·9 cm. 7. 8·0 cm.

Exercises. Page 291.

1. $\frac{1}{9}$. 2. 20 sq. ft. 3. 10 sq. cm. 4. 7 : 5. 5. 5·6″.

Exercises. Page 294.

2. 3·46″, 4·33″, 5·54″. 3. 9 ft. 3 in. 4. 3·75 sq. cm.
5 4·5 cm. 6. 15·48 sq. in. 7. 3·6 m. 1·5 m.
8. 90 acres. 9. 512 acres. 10. 1 cm. represents 15 metres.

Exercises. Page 295.

2. $1 : \sqrt{2}$. 6. $\frac{1}{64}$. 7. 256 : 81.

Exercises. Page 297.

3. 2·5 sq. cm., 6·4 sq. cm. 4. 4 : 1.
5. 7·2″. 8. 6·2 cm., 3·8 cm.

Exercises. Page 299.

1. $1 : \sqrt{2}$. 3. 4·6 cm. 4. 6·9 cm.

Exercises. Page 301.

1. 11·56 sq. in. 2. (i) 100, (ii) 576 units of area; (iii) 18·5.
3. (1·3, 2·4), (1·3, –2·4); 4·8; 2·4, nearly.

Exercises. Page 305.

10. (i) $10\frac{5}{6}''$ (ii) $15\frac{5}{8}$ ft.

Exercises. Page 313.

1. 15; (8, 6). 2. 15, 3·75; (0, 20), (0, 5). 3. 0·83″.
4. 1·17″. 6. 0·55″. 7. 0·93″.

Exercises. Page 317.

1. 15·3 sq. cm. 2. 8·55″. 3. 43·30 sq. cm.
4. 90°. 8. 6·83″; 1·17″. 10. 20·78 sq. cm., 20·8 cm.
12. 56°. 14. 1·2″; 22½°. 15. 63 ft.
16. 18 sq. cm. 17. 60½ sq. cm.

Exercises. Page 322.

1.

a	15°	30°	45°	60°	75°
PR	8·3	9·2	11·3	16·0	30·9

17·6 cm.; 25°.

2. (i) 8·9 sq. cm.; (ii) 72° or 108°; (iii) at right angles.
3. When the rod is equally inclined to the rulers.
4. When P is the mid-point of AB (i) is a maximum, (ii) is a minimum.
5. Minimum when $a=3$.
6. 45°. 7. 3·4″. 8. 9.

Exercises. Page 326.

2. (i) 6·0″; (ii) 1·2 cm.

Exercises. Page 329.

1. 4·0 cm., 8·8 cm. 2. (i) 1·20″; (ii) 1·93″.

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